Insurance: Mathematics and Economics 69 (2016) 156–167

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Minimizing lifetime poverty with a penalty for bankruptcy

Asaf Cohen*, Virginia R. Young

Department of Mathematics, University of Michigan, Ann Arbor, 48109, USA

ARTICLE INFO

Article history:
Received January 2016
Received in revised form
May 2016
Accepted 21 May 2016
Available online 26 May 2016

JEL classification:
C61
G02
G11

Keywords:
Poverty
Ruin
Occupation time
Optimal investment
Stochastic control

ABSTRACT

We provide investment advice for an individual who wishes to minimize her lifetime poverty, with a penalty for bankruptcy or *ruin*. We measure poverty via a non-negative, non-increasing function of (running) wealth. Thus, the lower wealth falls and the longer wealth stays low, the greater the penalty. This paper generalizes the problems of minimizing the probability of lifetime ruin and minimizing expected lifetime occupation, with the poverty function serving as a bridge between the two. To illustrate our model, we compute the optimal investment strategies for a specific poverty function and two consumption functions, and we prove some interesting properties of those investment strategies.
© 2016 Elsevier B.V. All rights reserved.

1. Introduction

In *Scarcity* (Mullainathan and Shafir, 2013) described how researchers have directly measured the reduction in mental capacity, or *bandwidth*, suffered by people who live with scarcity of money, time, or other resources. This so-called *bandwidth tax* is a result of how poverty forces one's mind to focus on dealing with lack of resources. In this paper, we provide investment advice for an individual who wishes to minimize her lifetime poverty, with a penalty for bankruptcy or *ruin*. We measure poverty via a non-negative, non-increasing function of (running) wealth. Thus, the lower wealth falls and the longer wealth stays low, the greater the penalty.

In most work concerning poverty, the goal is to measure how well or how poorly income or wealth is spread across a population.[1] In that literature, the focus is on the *distribution* of poverty across a group of individuals, not on controlling poverty for a given individual, as we do in this paper. In other words, our problem is one of micro-economics, not of macro-economics.

This paper is in the spirit of many of those in the collected works of Merton (1992), in that we optimize an objective function for an individual investing in a Black–Scholes market, that is, a market with one riskless asset earning interest at a constant rate and with one risky asset whose price follows geometric Brownian motion. Whereas the individual in Merton's model seeks to maximize expected utility of consumption and terminal wealth, the individual in our model minimizes expected "poverty", as measured by a non-decreasing function of (running) wealth, not of consumption or of terminal wealth. In our model, the individual's rate of consumption is given, but she chooses how to invest in order to minimize poverty during her lifetime.

We characterize the optimal investment policy by using the first and second derivatives of the value function (that is, the minimum expected poverty, with a penalty for ruin), which in turn is a solution of an (non-linear) ordinary differential equation. We prove comparative statistics of the value function for general poverty and consumption functions. Also, for a specific choice of the poverty function and two types of consumption functions, we compute (semi-)explicit expressions for both the value function and the optimal investment policy. For these special cases, we use the convex Legendre transform to determine the value function and the corresponding optimal investment policy.

Mathematically, our problem is closely related to those in the goal-seeking literature, such as minimizing the probability of

* Corresponding author.
E-mail addresses: shloshim@gmail.com (A. Cohen), vryoung@umich.edu (V.R. Young).
URLs: https://sites.google.com/site/asafcohentau/ (A. Cohen), http://dept.math.lsa.umich.edu/people/facultyDetail.php?uniqname=vryoung (V.R. Young).

[1] See Chakravarty (2009) and Lambert (2001) for extensive bibliographies in this line of research.

http://dx.doi.org/10.1016/j.insmatheco.2016.05.013
0167-6687/© 2016 Elsevier B.V. All rights reserved.

lifetime ruin,[2] maximizing the probability of reaching a bequest goal,[3] minimizing expected lifetime occupation (that is, the time that wealth stays below a given level),[4] or minimizing the probability of lifetime drawdown or expected lifetime spent in drawdown.[5] In fact, this paper generalizes the problems of minimizing the probability of lifetime ruin and minimizing expected lifetime occupation, with the poverty function serving as a bridge between the two, as we discuss in Remark 2.2 below.

The remainder of this paper is organized as follows. In Section 2, we describe the financial model, and we define the problem of minimizing the expectation of a non-negative, non-increasing function of wealth, the so-called *poverty function*, with a penalty for ruin. At the end of that section, we present a verification lemma that we use to solve the optimization problem. In Section 3, we prove some properties of the value function for a general poverty function, and in Section 4, we focus on a specific poverty function and two consumption functions. Section 5 concludes the paper.

2. The model

In Section 2.1, we present the financial market in which the individual invests, and we define the cost function that the individual wishes to minimize. Then, in Section 2.2, we present a verification lemma that we use to solve the individual's control problem.

2.1. Background and statement of problem

We study a model of an individual who trades continuously in a Black–Scholes market with no transaction costs. Borrowing and short selling are allowed. The market consists of two assets: a riskless asset and a risky asset. The price of the riskless asset follows the deterministic dynamics

$$dX_t = rX_t dt,$$

in which $r > 0$ is the constant riskless rate of return. The price of the risky asset follows geometric Brownian motion given by

$$dS_t = S_t\,(\mu dt + \sigma dB_t)\,,$$

in which $\mu > r, \sigma > 0$, and $(B_t)_{t\ge 0}$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$, in which $\mathcal{F}_t$ is the augmentation of $\sigma(B_u : 0 \le u \le t)$.

Let W_t denote the wealth of the individual's investment account at time $t \ge 0$. Let π_t denote the dollar amount invested in the risky asset at time $t \ge 0$. An investment policy $\{\pi_t\}_{t\ge 0}$ is *admissible* if it is an $\mathbb{F}$-progressively measurable process satisfying $\int_0^t \pi_s^2\, ds < \infty$ almost surely, for all $t \ge 0$.

We assume that the individual's net consumption rate equals $c(w) - A$, in which $c(w)$ is the rate of consumption when wealth equals w, and $A \ge 0$ is the constant rate of income. In Section 4, we assume that $c(w)$ is a continuous, non-decreasing function of wealth; in Section 4.2, we consider two specific consumption rates: a constant consumption rate $c(w) = c$, and a proportional consumption rate $c(w) = \kappa w$. Then, the wealth process follows the dynamics

$$\begin{cases} dW_t = [rW_t + (\mu - r)\pi_t - c(W_t) + A]\, dt + \sigma \pi_t dB_t, \\ \qquad t \ge 0, \\ W_0 = w. \end{cases}$$

[2] For an early reference, see Young (2004), and for a more recent reference, see Bayraktar and Zhang (2015).

[3] Bayraktar and Young (2015) and Bayraktar et al. (2016).

[4] Bayraktar and Young (2010).

[5] Chen et al. (2015), Angoshtari et al. (2015a), and Angoshtari et al. (2015b).

Let τ_a denote the first time that the wealth reaches a, which we will refer as the *ruin level*, that is, $\tau_a = \inf\{t \ge 0 : W_t \le a\}$. The individual wants to avoid living in poverty and to avoid bankruptcy or ruin during her lifetime. Let τ_d be the random time of death of the individual, independent of the Brownian motion driving the risky asset's price process. We assume that τ_d is exponentially distributed with hazard rate $\lambda > 0$, that is, $\mathbb{P}(\tau_d > t) = e^{-\lambda t}$.

Remark 2.1. Considering constant hazard rates simplifies the analysis and is essential for obtaining the main results of the paper. However, as we now explain, there is no significant loss of generality by assuming this. Moore and Young (2006) minimize the probability of lifetime ruin with varying hazard rate. They provide a scheme that closely approximates the minimum probability of ruin in the case for which the true hazard rate is Gompertz. The scheme is as follows: once a year the individual calculates the inverse of her life expectancy at that time. Then, she sets the hazard rate equal to this inverse during the year and applies the optimal investment strategy, as given for a constant hazard rate, during the year. Thus, considering a constant hazard rate and updating it each year is not too restrictive.

Also, in the setting of an endowment fund of an organization, it is not unreasonable to assume that the hazard rate for the organization is constant. There is no reason that an organization will "age".

The individual seeks to minimize the following cost over admissible investment strategies.

$$\begin{aligned} J(w; \{\pi_t\}) &:= \mathbb{E}^w \left[\int_0^{\tau_a \wedge \tau_d} l(W_t)\, dt + \rho \cdot \mathbf{1}_{\{\tau_a \le \tau_d\}} \right] \\ &= \mathbb{E}^w \left[\int_0^\infty \lambda e^{-\lambda t} \left(\int_0^{\tau_a \wedge t} l(W_s)\, ds \right) dt + \rho\, e^{-\lambda \tau_a} \right] \\ &= \mathbb{E}^w \left[\int_0^{\tau_a} e^{-\lambda t}\, l(W_t)\, dt + \rho\, e^{-\lambda \tau_a} \right], \end{aligned} \tag{2.1}$$

in which $\mathbb{E}^w$ denotes expectation conditional on $W_0 = w$, $l(\cdot)$ is a non-negative, non-increasing function that measures the economic and physical costs of living in poverty, and $\rho > 0$ is a constant penalty for lifetime ruin. We call $l(\cdot)$ the *poverty function*. If we were to allow $\rho < \frac{l(a+)}{\lambda}$, then the individual might find it advantageous to commit financial suicide by allowing her wealth to fall to the ruin level instead of continuing to live in poverty. Therefore, to prevent financial suicide, we assume that $\rho \ge \frac{l(a+)}{\lambda}$ throughout this paper. One can interpret the difference $\rho - \frac{l(a+)}{\lambda}$ as the *net* penalty for ruin, that is, net of the penalty for ruining instead of remaining in poverty (near a) for the rest of one's life. Furthermore, we assume that $l(a+) > 0$; otherwise, $l(\cdot) \equiv 0$, and our problem would be equivalent to minimizing the probability of lifetime ruin.

The function V defined by

$$V(w) := \inf_{\{\pi_t\}} J(w; \{\pi_t\}) \tag{2.2}$$

is the *value function*, in which we minimize over admissible investment strategies.

Remark 2.2. In Bayraktar and Young (2010), a special case of the problem in this paper is studied, the so-called *lifetime occupation* problem. They minimize the expected time that wealth spends below 0, subject to the "game" ending if wealth falls below some very low level, $-L$ in Bayraktar and Young (2010). Thus, if one sets $\rho = 1/\lambda$, $l(w) = \mathbf{1}_{\{w<0\}}$, and $a = -L$, then V in (2.2) plus the pre-existing time spent below 0 equals the minimum lifetime occupation as defined in Bayraktar and Young (2010). Note that by setting $l(w) = \mathbf{1}_{\{w\le 0\}}$, we measure the running time that wealth spends below 0, and by setting $\rho = 1/\lambda$, we assume that once

Download English Version:

https://daneshyari.com/en/article/5076369

Download Persian Version:

https://daneshyari.com/article/5076369

Daneshyari.com