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A simple compound scan statistic useful for modeling insurance and risk management problems

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ABSTRACT

In the present work we study the distribution of a random sum of random variables which is related to a binary scan statistic for Markov dependent trials. The motivation of the model studied herein stems from several areas of applied science such as actuarial science, financial risk management, quality control and reliability, educational psychology, engineering, etc.

Let us consider a sequence of binary success/failure trials and denote by T_k the waiting time for the first occurrence of two successes separated by at most k failures, where $k \geq 0$ is any integer. Let also $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed (i.i.d) discrete random variables, independent of T_k. In the present article we develop some results for the distribution of the compound random variable $S_k = \sum_{t=1}^{T_k} Y_t$ and illustrate how these results can be profitably used to study models pertaining to actuarial science and financial risk management practice.

1. Introduction

Many problems encountered in a large number of applied areas, may be modeled by the aid of dichotomous (binary) variables taking on the values 1 (success, S) or 0 (failure, F). A popular random variable associated with sequences of binary outcomes, is the number of trials till a predetermined criterion is satisfied (stopping rule). In the classical model, the time between successive trials is not taken into account; however, in most real-life situations, the time between consecutive trials varies and therefore should be treated as a random variable as well.

Motivated by the aforementioned observation, in the present work we study a probability model with random times between consecutive trials, and elucidate how it can be implemented in problems arising in actuarial science and financial risk management practice. Before proceeding to the formal definition and analysis of the stochastic model we are going to deal with, we present two specific examples emerging in the financial and insurance industry which can be accommodated in the framework we are focusing on.

1.1. Insurance portfolio surveillance

Insurance industry has a long-standing relationship with risk. The definition of the actuarial profession by the Institute of Actuaries and the Faculty of Actuaries clearly states that it is the responsibility of an actuary to manage assets and liabilities by analyzing past events, assessing the present risk and modeling what could happen in the future. One of the principal functions of risk management in the insurance industry is to determine the amount of capital a company needs to hold as a buffer against unexpected future losses on its portfolio.

In the classical aggregate loss model used in insurance analytics, the interest focuses on the total loss amount over a fixed time period $[0, t]$. Apparently the number of losses over that period is a random variable, say $N(t)$, and the individual losses $X_1, X_2, \ldots$ should also be treated as random variables. The aggregate loss can then be expressed as a random sum of the form $S_t = \sum_{i=1}^{N(t)} X_i$.

Departing from the classical setup, let us look at a framework that can be used by an insurance company to create a surveillance mechanism over a specific portfolio it holds. Let us denote by $X_1, X_2, \ldots$ the individual claim sizes arriving at the company. An unexpected future loss, defined by the aid of a threshold x_0, could be used as a warning for an approaching high risk situation for the company.

Apparently, the series of outcomes generated over time by the aforementioned process may be modeled by the aid of a

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sequence of binary variables $\xi_1, \xi_2, \ldots$ taking on the values 1 (success, S) or 0 (failure, F) where 1 designates an unexpected loss (i.e. a loss exceeding x_0) and 0 a loss that is smaller than x_0. Assuming independence between successive claims, the sequence $\xi_1, \xi_2, \ldots$ turns out to be a sequence of Bernoulli trials with success probability $p = P[\xi_t = 1] = P[X_t > x_0]$ and failure probability $q = P[\xi_t = 0] = P[X_t \le x_0]$. A plausible criterion to confer that a high risk situation is approaching for the insurance company would be to observe at least two successes (unexpected losses) that are very close to each other, e.g. when an outcome of the form SS, SFS, $SFFS$ occurs.

The aforementioned model can be accommodated in the next more general setup. Let $\xi_1, \xi_2, \ldots$ be an infinite sequence of binary outcomes and denote by T_k the waiting time for the first occurrence of two successes separated by at most k failures (k is a non-negative integer). Clearly, T_k counts the number of trials required to observe for the first time one of the patterns $SS,\ SFS, \ldots, S\overbrace{F\cdots F}^{k}S$. Let us now denote by Y_1 the time when the first claim arrives and by Y_t the interarrival time between the $(t-1)$th and tth claims ($t \ge 2$). Manifestly, the times when the unexpected losses occur are random variables continuous or discrete (when a specific time unit is used, e.g. day, month, etc.). Then the total time till a high risk signal is created for the portfolio (i.e. because of the occurrence of unexpected losses being very close to each other) will be described by the random variable $S_k = \sum_{t=1}^{T_k} Y_t$.

1.2. A financial risk management model

Assume that a bank is subject to a sequence of stress tests over time. Using several indices related to the bank's economical health, the bank may be classified as appropriately functioning (low risk of defaulting) or not. For example, it is widely recognized that a bank's capitalization is of utmost importance and provides the main line of defense for absorbing unexpected losses; therefore it may be used as an important risk measure signaling an oncoming credit event, i.e. default. A measure of bank capitalization health is provided by the Capital Ratio (*CAR*), which is defined as (Tier1 + Tier2)/Risk-Weighted Assets. For regulatory purposes, the Basel Accord has adopted a simple dichotomous classification that characterizes a bank either as undercapitalized or not, depending on whether its *CAR* falls below or above 8%; see e.g. Demstez et al. (1996), Flannery and Sorescu (1996), Estrella et al. (2000), Goldberg and Hudgins (2002), Lindquist (2004), Berger et al. (2008) and Koutras and Drakos (2013).

Manifestly, the series of stress tests outcomes can be modeled by the aid of a sequence of binary variables $\xi_1, \xi_2, \ldots$ taking on the values 1 (success, S) or 0 (failure, F) where 1 indicates an undercapitalized bank (i.e. a bank that failed in the stress test) and 0 a well-capitalized one. The sequence $\xi_1, \xi_2, \ldots$ may arise as follows. Let X_t denote the *CAR* of the bank subject to the stress test at time t, $t = 1, 2, \ldots$. Adopting the Basel Accord dichotomous classification we may consider that the bank failed the stress test if $X_t < 8\%$ and therefore $p = P[\xi_t = 1] = P[X_t < 0.08]$ and $q = P[\xi_t = 0] = P[X_t \ge 0.08]$.

However, in this case the independence assumption made before for the sequence of binary outcomes cannot be substantiated, since the values of ξ_t are determined by X_t, $t = 1, 2, \ldots$ which in practice are not independent. In most cases, using past data, one may easily verify that the stochastic behavior of a bank's *CAR* (evolution of the X_t's, $t = 1, 2, \ldots$) can be adequately described by a first order Markov dependent process. This is readily substantiated by assuming that X_t depends on the magnitude of the previous capitalization X_{t-1}, an assumption which is quite realistic. In this case, the outcomes ξ_t, $t = 1, 2, \ldots$. of the stress tests form a sequence of time-homogeneous two-state $(0-1)$ Markov dependent trials.

Exploiting the same arguments as in the insurance portfolio surveillance example, we may consider that a bank is susceptible to default if it fails in two stress tests that are very close to each other. Let us also assume that the times of the stress tests are random; such a scenario seems reasonable in the case where we wish randomness to prevent the bank from having information on the stress test times. Denoting by Y_1 the first stress test time and by Y_t the interarrival time between the $(t-1)$th and tth stress tests ($t \ge 2$), the total time till a default signal is created for the monitored bank will be described by the random variable $S_k = \sum_{t=1}^{T_k} Y_t$.

In the present article, we study the distribution of the statistic $S_k = \sum_{t=1}^{T_k} Y_t$ arising in the previous examples. The random variables ξ_t, $t = 1, 2, \ldots$ which give birth to the enumerating random variable T_k are assumed to be a sequence of time-homogeneous two-state Markov dependent trials; the respective results for the i.i.d. case are also derived as a special case of the Markov dependence model. In Section 2 we introduce the definitions and notations that will be used throughout the paper. In Section 3 we give some results for the probability generating functions (pgf), the moment generating functions (mgf) and the moments of the compound scan statistic $S_k = \sum_{t=1}^{T_k} Y_t$. Section 4 addresses the problem of evaluating the probability mass function (pmf) of S_k in the case where Y_t's are discrete random variables. We indicate how one can establish effective recursive schemes for the pmf of S_k and discuss a method that leads to non-recursive schemes. Finally, in Section 5 we present some numerical results and illustrate how they can be practically used in real-life applications.

2. Definitions, notations and preliminary material

Let $\xi_1, \xi_2, \ldots$ be an infinite sequence of binary outcomes and denote by T_k the waiting time for the first occurrence of two successes which are separated by at most k failures. Manifestly, T_k counts the number of trials required to observe for the first time one of the patterns $SS,\ SFS, \ldots, S\overbrace{F\cdots F}^{k}S$.

The random variable T_k is a special case of a scan statistic, see e.g. Boutsikas and Koutras (2002, 2006), Chen and Glaz (1997), Fu et al. (2012), Glaz (1983), Glaz and Naus (1991), Greenberg (1970), Koutras (1996), Koutras and Alexandrou (1995), Saperstein (1973), Wu et al. (2013) or the excellent monograph by Glaz et al. (2009). Note that, for $k = 0$ the random variable T_k is enumerating success runs of length 2 and therefore it follows a geometric distribution of order 2; the interested reader may consult the book by Balakrishnan and Koutras (2002) for more details and results relating to waiting times for runs and scans.

Let us next define another sequence of random variables (either discrete or continuous) $Y_1, Y_2, \ldots$, that are positive valued, independent and identically distributed and independent of the waiting time random variable T_k, as well. Let Y_1 be the waiting time for the first occurrence of the event of interest, e.g. the arrival of a claim at an insurance company or the implementation of a stress test for a bank, etc.; in addition denote by Y_t the interarrival time between the $(t-1)$th and tth occurrences of that event ($t \ge 2$). In the sequel we shall proceed to a detailed study of the distribution of the random variable $S_k = \sum_{t=1}^{T_k} Y_t$ which represents the total time till the first occurrence of two successes which are separated by at most k failures. Since S_k is a random sum of random variables and the number of summands is determined by a (simple) scan statistic, we shall refer to in by the term *compound scan statistic*.

In the next sections we shall present several results for the distribution of S_k under the assumption that the binary outcomes ξ_t, $t = 1, 2, \ldots$ form a sequence of time-homogeneous two-state $(0-1)$ Markov dependent trials with $P[\xi_t = j | \xi_{t-1} = i] = p_{ij}$ for