Insurance: Mathematics and Economics 64 (2015) 126–134

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

A bivariate risk model with mutual deficit coverage

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ARTICLE INFO

Article history:
Received January 2015
Received in revised form
May 2015
Accepted 7 May 2015
Available online 22 May 2015

Keywords:
Two-dimensional risk model
Survival probability
Coupled processor model
Wiener–Hopf factorization
Surplus note
Mutual insurance

ABSTRACT

We consider a bivariate Cramér–Lundberg-type risk reserve process with the special feature that each insurance company agrees to cover the deficit of the other. It is assumed that the capital transfers between the companies are instantaneous and incur a certain proportional cost, and that ruin occurs when neither company can cover the deficit of the other. We study the survival probability as a function of initial capitals and express its bivariate transform through two univariate boundary transforms, where one of the initial capitals is fixed at 0. We identify these boundary transforms in the case when claims arriving at each company form two independent processes. The expressions are in terms of Wiener–Hopf factors associated to two auxiliary compound Poisson processes. The case of non-mutual agreement is also considered. The proposed model shares some features of a contingent surplus note instrument and may be of interest in the context of crisis management.

1. Introduction

Insurance companies cannot operate in isolation from financial markets or from other insurance and reinsurance companies. Hence it is important to understand the effect of interaction on the main characteristics of an insurance company. Multivariate risk models, however, present a serious mathematical challenge with few explicit results up to date, see Asmussen and Albrecher (2010, Chapter XIII.9). This paper focuses on a nonstandard, but rather general bivariate risk model and provides an exact analytic study of the corresponding survival probability, borrowing some ideas from the analysis of a somewhat related queueing problem in Boxma and Ivanovs (2013). The proposed model may be of interest in the context of crisis management due to its relation to contingent surplus notes and mutual insurance. Moreover, it allows for an explicit structural result without imposing overly strict assumptions such as proportional or dominating claims, see Section 1.1.

We consider a bivariate Cramér–Lundberg risk process as a model of surplus of two insurance companies (or two lines of one insurance business). The special feature of our model is that the companies have a mutual agreement to cover the deficit of each other. More precisely, if company 1 gets ruined, with its capital decreasing to a value $-x < 0$, then company 2 compensates this deficit, bringing the capital of company 1 back to 0. However, this comes at a price; a unit of capital received by company 1 requires $r_1 \geq 1$ from company 2 (cf. Fig. 1). If this would cause the capital of company 2 to go below 0, then both companies are said to be ruined. Similarly, if company 2 gets a deficit $-y < 0$, company 1 compensates this deficit, but its capital reduces by $r_2 y \geq y$; and if this would cause the capital of company 1 to go below 0, then again both companies are said to be ruined. Finally, ruin may also be caused by a single event bringing the surplus processes of both companies below 0. It may be more realistic to assume that if one company cannot save the other from ruin then it does not transfer any capital at that instant and continues to operate. Note, however, that survival of both companies in this set-up corresponds to our previous notion of survival.

Our main goal in this study is to provide an exact analysis of the (Laplace transform of the) probability of survival (i.e., ruin never occurs) $\phi(u, v)$, as a function of the vector (u, v) of initial capitals. We do this by (i) expressing the two-dimensional Laplace transform $F(s_1, s_2)$ of $\phi(u, v)$ in terms of the transforms of $\phi(u, 0)$ and $\phi(0, v)$, and (ii) determining the latter two transforms by solving a Wiener–Hopf boundary value problem in the case of independent claim streams. In the latter step, a key role is played by the Wiener–Hopf factorization of two auxiliary compound Poisson processes.

In our terminology, if one company cannot save the other then both are declared ruined, whereas in practice it is more likely that the company with a positive capital pays out all it has, but then

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http://dx.doi.org/10.1016/j.insmatheco.2015.05.006
0167-6687/

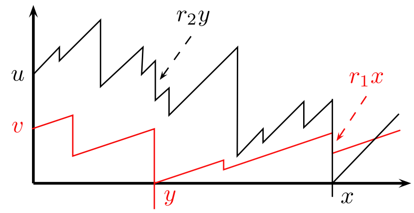

Fig. 1. Illustration of mutual deficit coverage (no common shocks).

continues its operation. Hence it may be more appropriate to call ϕ the probability of survival of *both* companies, or the probability that external help is never needed, see Remark 3. One may also notice that it is clearly better (with respect to survival) to merge the two lines eliminating transaction costs. There may be cases, however, where a merger is not possible due to legal, regulatory or other issues. Outside of an insurance context, the two lines may be two separate physical entities such as water reservoirs or energy sources. Furthermore, one may consider the case where $r_i < 1$ for at least one company, see Remark 2, so that merging may not be optimal. This may correspond to the case where part of the deficit is written off or is covered by some other fund.

Our model resembles two insurance companies with a mutual insurance fund, which is used to cover the deficit of each of them. In our case, however, there is no separate fund—it is the other company which provides the capital to cover the deficit. Another related notion is that of a contingent surplus note which is a form of a CoCo bond issued by an insurance company. This bond pays a higher coupon, because of the risk that it is converted into surplus if a trigger event (e.g. deficit) occurs. Finally, we may consider an insurance company and a governmental fund which is used to save companies from default. In this case capital transfers are only possible from the fund to the company; this particular scenario is discussed in Section 7.

The above-sketched risk model bears some resemblance to a two-dimensional queueing model of two coupled processors. This model features two $M/G/1$ queues, each of which, in isolation, is known to be the dual of a Cramér–Lundberg insurance risk model, cf. Asmussen and Albrecher (2010, Chapter I.4a). Just like in our risk model, the two processors are coupled by the agreement to help each other. When one of the two $M/G/1$ queues becomes empty, the service speed of the other server – say, server i – increases from 1 to $r_i \geq 1$, i.e. server i is being helped by the idle server. It should be stressed that this similarity is rather loose and that there is no clear duality relation between these bivariate risk and queueing models. Strikingly, the crucial ideas of the analysis of the coupled processor model in Boxma and Ivanovs (2013) apply to our present setup as well. Moreover, solutions to both problems are based on the same Wiener–Hopf factors, which hints that there might be a certain duality relation between the two.

1.1. Related literature

Despite their obvious relevance, exact analytic studies of multidimensional risk reserve processes are scarce in the insurance literature. A special, important case is the setting of proportional reinsurance, which was studied in Avram et al. (2008). There it is assumed that there is a single arrival process, and the claims are proportionally split among two reserves. In this case, the two-dimensional exit (ruin) problem becomes a *one*-dimensional first-passage problem above a piece-wise linear barrier. Badescu et al. (2011) have extended this model by allowing a dedicated arrival stream of claims into only one of the insurance lines. They show that the transform of the time to ruin of at least one of the reserve processes can be derived using similar ideas as in Avram et al. (2008).

An early attempt to assess multivariate risk measures can be found in Sundt (1999), where multivariate Panjer recursions are developed which are then used to compute the distribution of the aggregate claim process, assuming simultaneous claim events and discrete claim sizes. Other approaches are deriving integro-differential equations for the various measures of risk and then iterating these equations to find numerical approximations (Chan et al., 2003; Gong et al., 2012), or computing bounds for the different types of ruin probabilities that can occur in a setting where more than one insurance line is considered, see Cai and Li (2005, 2007). In an attempt to solve the integro-differential equations that arise from such models, Chan et al. (2003) derive a Riemann–Hilbert boundary value problem for the bivariate Laplace transform of the joint survival function (see Badila et al., 2014 for details about such problems arising in the context of risk and queueing theory and the book (Cohen and Boxma, 1983) for an extended analysis of similar models in queueing). However, this functional equation for the Laplace transform is not solved in Chan et al. (2003). In Badila et al. (2014) a similar functional equation is taken as a departure point, and it is explained how one can find transforms of ruin related performance measures via solutions of the above mentioned boundary value problems. It is also shown that the boundary value problem has an explicit solution in terms of transforms, if the claim sizes are ordered. In Badila et al. (2015) this is generalized to the case in which the claim amounts are also correlated with the time elapsed since the previous claim arrival.

Bivariate models where one company can transfer its capital to the other have already been considered in the literature. Recently, Avram and Pistorius (in preparation) proposed a model of an insurance company which splits its premiums between a reinsurance/investment fund and a reserves fund necessary for paying claims. In their setting only the second fund receives claims, and hence all capital transfers are one way: from the first fund to the second. Another example is a capital-exchange agreement from Lautscham (2013, Chapter 4), where two insurers pay dividends according to a barrier strategy and the dividends of one insurer are transferred to the other unless the other is also fully capitalized. This work resulted in systems of integro-differential equations for the expected time of ruin and expected discounted dividends, which are hard to solve even in the case of exponential claims.

Finally, we briefly list related contributions in the queueing context. The joint queue length distribution of the coupled processor model has been derived by Fayolle and Iasnogorodski (1979), in the case that the service time distributions at both queues are exponential. In their pioneering paper, they showed how the generating function of the joint steady-state queue length distribution can be obtained by solving a Riemann–Hilbert boundary value problem. Cohen and Boxma (1983) generalized this queueing model by allowing general service time distributions. They obtained the Laplace–Stieltjes transform of the joint steady-state workload distribution by solving a Wiener–Hopf boundary value problem. In Boxma and Ivanovs (2013) the model of Cohen and Boxma (1983) was extended by considering a pair of coupled queues driven by independent spectrally positive Lévy processes and a compact solution was obtained. There the model was also linked to a two-server fluid network.

1.2. Organization of the paper

In Section 2 we describe the model in detail. In Section 3 we derive an integral equation for the survival probability $\phi(u, v)$, as a function of the vector (u, v) of initial amounts of capital. Section 4 is devoted to the derivation of a so-called kernel equation for the two-dimensional Laplace transform of $\phi(u, v)$, see Proposition 1. After a brief discussion of the net profit condition, in Section 5, we solve the kernel equation in the case of independent claim streams