Insurance: Mathematics and Economics 64 (2015) 151–161

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal dynamic asset allocation of pension fund in mortality and salary risks framework

Zongxia Liang, Ming Ma*

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

HIGHLIGHTS

- Study optimal asset allocation of pension fund with mortality and salary risks.
- The market is a combination of financial market and incomplete insurance market.
- The closed-forms of the approximate optimal investment policy are derived.
- Investigate the efficiency of the approximation.
- Solve an optimal ALM problem with mortality risk and salary risk under CRRA utility.

ARTICLE INFO

Article history:
Received July 2014
Received in revised form
April 2015
Accepted 20 May 2015
Available online 30 May 2015

JEL classification:
G11
G22
G32
C61

MSC:
91G10
91B16
93E20
91B30

Submission Classifications:
IE13
IE12
IE43
IB81
IB52
IE53

Keywords:
Assets liabilities management (ALM)
Optimal dynamic asset allocation
Mortality risk
Salary risk
Incomplete market
Stochastic dynamic programming
Martingale method

ABSTRACT

In this paper, we consider the optimal dynamic asset allocation of pension fund with mortality risk and salary risk. The managers of the pension fund try to find the optimal investment policy (optimal asset allocation) to maximize the expected utility of terminal wealth. The market is a combination of financial market and insurance market. The financial market consists of three assets: cashes with stochastic interest rate, stocks and rolling bonds, while the insurance market consists of mortality risk and salary risk. These two non-hedging risks cause incompleteness of the market. By martingale method and dynamic programming principle we first derive the approximate optimal investment policy to overcome the difficulty, then investigate the efficiency of the approximation. Finally, we solve an optimal assets liabilities management(ALM) problem with mortality risk and salary risk under CRRA utility, and reveal the influence of these two risks on the optimal investment policy by numerical illustration.

1. Introduction

As a major factor of social security system, the pension funds now are definitely confronted with many serious risks, such as

* Corresponding author.
E-mail addresses: zliang@math.tsinghua.edu.cn (Z. Liang), maming0292@gmail.com (M. Ma).

http://dx.doi.org/10.1016/j.insmatheco.2015.05.008

finance risk and mortality risk as well as salary risk during the diversification process suiting the demands in the market. The assets liabilities management (ALM) for a company has become a vital important act to control these different threats. The ALM theory is pioneered by Anglo-Saxon financial institutions during the 1970s to study the mismatches between the assets and liabilities. There are two fundamental methods to solve ALM problem, i.e., the so-called stochastic dynamic programming and martingale methods. The stochastic dynamic programming method is first proposed by Merton (1971) and relies on the stochastic control and theory of Hamilton–Jacobi–Bellman (HJB) equations. Recently, Menoncin and Scaillet (2006) applied this method to life annuity. By using the theory of Lagrange multipliers, the martingale method was beautifully developed by Cox and Huang (1989) in the setting of complete market. Some successful applications in ALM problem without mortality risk can be referred to Boulier et al. (2001) and Deelstra et al. (2003, 2004).

It is well-known that one can get the same solution for one optimization problem by these two methods in the setting of complete market but it is not sure in the setting of incomplete market. Hainaut and Devolder (2006) showed that these two methods also get the same solution on the same optimization problem in the setting of incomplete market for a pure endowments insurance case. Whereafter, Hainaut and Devolder (2007) successfully solved an optimal ALM problem with mortality risk and dividends in incomplete market by stochastic dynamic programming. The solution obtained in incomplete market is indeed an approximation of the target process by a projection from the expanded space, thus it is not the exact optimal solution in self-financed space and so one has great doubt about efficiency of the approximation.

However, to the best of our knowledge in incomplete markets, no literature studies the exact optimal solution in self-financed space and how to get it, so the approximation is optimal in some sense as soon as its efficiency is confirmed. To demonstrate the efficiency of approximation, we first consider a degenerated ALM problem whose exact optimal solution can be easily obtained and compare exact solution with approximation by numerical illustration. Then, we solve the ultimate ALM problem with two non-hedging risks to be more close to real market by efficient approximation.

The first non-hedging risk considered in this paper is the mortality intensity risk. It is influential in a long time insurance contract and makes the mortality risk become a systematic risk. The mortality risk is first described by a doubly stochastic counting process in Brémaud (1981) and the related models were also established by the comparison of the mortality risk with credit risk in financial market. In particular, the total number of death is a Poisson process with stochastic intensity. For describing stochastic mortality intensity better, the affine models are widely expanded, including the Ornstein–Uhlenbeck (OU) process and CIR models, which have been both minutely studied by Luciano et al. (2012). In this paper we will use the OU model to describe mortality intensity because it has an explicit expression. But when the actuarial one turns to be stochastic, the interplay of stochastic intensity between assets and liabilities causes the difficulty of calculating further liabilities. We calculate the explicit expressions of interactional terms and use the property of multidimensional Gaussian processes to get over this difficult.

The second non-hedging risk investigated in this paper is the salary risk because some advanced defined benefit (DB) pension plans are related to employees' salaries of retired time. These kinds of pension plans develop rapidly because they can promote employees' initiative in working and guarantee the same living standards after retiring. In this paper, we also introduce a risk factor to describe the working atmosphere in a department, which is indispensable in salary besides financial market. Following this motivation, Blake et al. (2001) first studied the pension plans related to the salary risk. Cairns et al. (2006) studied the optimal dynamic asset allocation for defined contribution pension plans. The reader also refers to Hainaut and Deelstra (2011), Guan and Liang (2014), He and Liang (2013a,b) for the recent works about defined contribution pension plans. With the same idea, the salary risk is described by an exponential Brownian motion in this paper, while the non-hedging Brownian motion describing the initiative of employees.

The rest of the paper is organized as follows. Sections 2 and 4 present the mathematical models of assets and actuarial liabilities, respectively. Section 3 establishes two non-hedging risk models related to the ALM problems. Section 5 formulates the optimization problem and derives its general solution by Lagrange multipliers. Section 6 compares the optimal strategies in self-financed space with the approximation obtained by dynamic programming, and demonstrates the efficiency of this approximation. In Section 7, we first get the closed-form of the optimal solution for the optimization problem with the two non-hedging risks under CRRA utility, which is more qualified to be a utility function than CARA. Then we give a numerical illustration to show how the economic behaviors of mortality and salary risks impact on the optimal strategies. Finally, we point out the essence of salary risk and mortality risk. The last section is a conclusion.

2. Assets

We consider a complete financial market composed of three assets: cashes with stochastic interest rate, stocks and rolling bonds. The financial probability space is denoted by $(\Omega^f, \mathcal{F}^f_\infty, \mathcal{F}^f, P^f)$, the filtration $\mathcal{F}^f = \{\mathcal{F}^f_t\}_{t\geq 0}$ is generated by a two-dimensional Brownian motion $\{W^f_t\} = \{(W^s_t, W^r_t)\}$, where the Brownian motions $\{W^s_t\}$ and $\{W^r_t\}$ on $(\Omega^f, \mathcal{F}^f_\infty, P^f)$ are independent, and $\mathcal{F}^f_\infty = \sigma(\bigcup_{t\geq 0}\mathcal{F}^f_t)$.

The existence of a unique equivalent measure Q^f is guaranteed by the completeness of the financial market. Under these assumptions we characterize the dynamics of the three assets as follows: The stochastic interest rate is modeled by the following Vasicek's model:

$$dr_t = a_r(\bar{r} - r_t)dt + \sigma_r dW^r_t = a_r\left(\bar{r} - r_t - \sigma_r\frac{\theta_r}{a_r}\right)dt + \sigma_r\underbrace{(dW^r_t + \theta_r dt)}_{d\widetilde{W}^r_t}, \tag{2.1}$$

where the $\{\widetilde{W}^r_t\}$ is a Brownian motion under the Q^f. The parameters a, $\bar{r}$ and σ_r are positive constants but the θ_r is a negative constant.

The rolling bond $\{R^K_t\}$ with maturity K is defined by the following SDE:

$$\frac{dR^K_t}{R^K_t} = r_t dt - \sigma_r n(K)(dW^r_t + \theta_r dt) = r_t dt - \sigma_r n(K) d\widetilde{W}^r_t. \tag{2.2}$$

The risk premium of the rolling bond is constant $\nu_r = -\sigma_r n(K)\theta_r$, where the $n(K)$ is determined by the maturity of the rolling bond: $n(K) = \frac{1}{a_r}(1 - e^{-a_r K})$.

The stock $\{S_t\}$ is a geometric Brownian motion satisfying the following SDE:

$$\frac{dS_t}{S_t} = r_t dt + \sigma_{sr}(dW^r_t + \theta_r dt) + \sigma_s(dW^s_t + \theta_s dt) = r_t dt + \sigma_{sr} d\widetilde{W}^r_t + \sigma_s d\widetilde{W}^s_t, \tag{2.3}$$

where the σ_{sr}, σ_s and θ_s are positive constants and the stock's risk premium is $\nu_s = \sigma_{sr}\theta_r + \sigma_s\theta_s$.