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Less is more: Increasing retirement gains by using an upside terminal wealth constraint

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HIGHLIGHTS

- We constrain the terminal wealth to be below a target value.
- Maximize the expected utility of power utility subject to the constraint.
- Solve to find the optimal investment strategy.
- The lower quantiles and the probability of reaching the target value are increased.
- May be attractive to investors who are averse to poor financial outcomes.

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ABSTRACT

We solve a portfolio selection problem of an investor with a deterministic savings plan who aims to have a target wealth value at retirement. The investor is an expected power utility-maximizer. The target wealth value is the maximum wealth that the investor can have at retirement.

By constraining the investor to have no more than the target wealth at retirement, we find that the lower quantiles of the terminal wealth distribution increase, so the risk of poor financial outcomes is reduced. The drawback of the optimal strategy is that the possibility of gains above the target wealth is eliminated.

1. Introduction

Investing for retirement is usually characterized by a period of savings followed by a period of consumption. The question of how to invest the savings before retirement has been considered widely in the academic literature. We consider the problem of how to invest an initial wealth and periodic amounts in order to reach some target capital at a fixed time horizon that represents the intended retirement date. This is a different formulation of one of the problems described in Dhaene et al. (2005, p. 277), in which an investor wishes to find the optimal constant-proportion portfolio that attains the highest target capital with a fixed probability. We constrain the investor to have *at most* the target capital at the time of retirement, whereas Dhaene et al. (2005) ensure that at least the target capital is attained with maximum probability. Since our focus is on a broad analysis of following the optimal strategy, we assume throughout this paper a simple continuous-time complete market model. Wealth can be invested in a risky asset and in a risk-free asset. Our discussion is about the strategies regarding the amount invested in each of those. As the investment period is long, we are interested in the long-run outcome, namely the distribution

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of the terminal wealth, rather than in the fluctuations of wealth during the savings phase.

Our paper is about the reduction of the risk of terminal wealth being too large and too low. We assume that investors are willing to accept that gains may not be too large in the long-run, if there is a higher chance that terminal wealth is not too low. We consider this problem to be of crucial importance to consumers, who do not want the accumulated value of their retirement savings to be insufficient for their retirement needs. Our approach differs from Gerrard et al. (2014) who examined the lowest part of the terminal wealth distribution after savings and consumption. Here we study only the savings phase and we rather fix an upper target wealth, which should not be exceeded at the terminal time point. This is what we call a constrained strategy. As a return for the sacrifice of profits, the terminal wealth distribution is more concentrated in the values below the target wealth than in the constrained plan, so that the probability of small values is lower than under the pure unconstrained investment strategy.

We find an optimal strategy for investors in the current framework. This result follows from maximizing the expected utility of terminal wealth plus designing a call option on the fixed target. Moreover, we also find that there is an optimal target level of wealth to be chosen, which provides a larger difference in the rate of return to the investor compared with the optimal unconstrained strategy, in which the investor holds a fixed proportion of his wealth in the risky asset.

We do not look at portfolio selection when there is more than one risky asset available, like in Van Weert et al. (2010), but we do take into consideration risk aversion through a utility function in addition to the constraint on the terminal wealth values. We also permit a dynamic asset allocation strategy. Our results can easily be extended to the case where investors have both an upper and a lower target in the terminal wealth distribution.

We should mention here several recent works on dynamic asset allocation strategies. Some authors do not formally specify the investor's problem and simply propose an investment strategy. Basu et al. (2011) look at performance relative to a target return and suggest a contrarian strategy of switching the investor's asset allocation between 100% of wealth invested in stocks, and 80% of wealth in stocks and the remainder in bonds according to whether the cumulative target return is attained or not. Similar strategies are compared in Basu and Drew (2009). Both papers show that defensiveness towards the end of the investor's time horizon, through diminishing the investment in the risky assets (a so-called lifecycle strategy), is costly in terms of the overall return (Guillén et al., 2013 arrives at a similar conclusion using a different methodology).

Another approach in the literature is to specify the investor's problem within a model and then determine the optimal investment strategy. Typically, the investor's core problem is to maximize the expected utility of terminal wealth subject to specified constraints being satisfied. Grossman and Zhou (1996) impose the constraint that the terminal wealth must be at least some fraction of the initial wealth. Korn and Trautmann (1995) impose a constraint on the expected value of the terminal wealth. Other authors impose the constraint that the investor's terminal wealth is at least a minimum value with a certain probability. (This is similar to the problem in Dhaene et al. (2005, p. 277) except that the latter maximize the minimum value directly and do not use a utility function.) In Boyle and Tian (2007), the minimum value is a random variable that models a benchmark strategy. Bouchard et al. (2010) prove a viscosity solution characterization of the value function in a very general setting when there are terminal wealth constraints. De Franco and Tankov (2011) and Gaibh et al. (2009) use a risk-measure constraint that is applied only to terminal losses that are worse than a fixed level.

Browne (1999) solves a similar problem to the one that we consider, except that he maximizes directly the probability of reaching the target retirement wealth. The formulation is attractive, since it requires only a target wealth to be specified by the investor; calibrating utility functions to individual investors is complicated (von Gaudecker et al., 2011). In other words, Browne (1999) does not capture explicitly the investor's emotional responses to investment gains and losses, as we do here very simply with a power utility function or as Jin and Zhou (2008) do by applying prospect theory. However, the consequence is that following the optimal strategy in Browne (1999) results in an "all-or-nothing" terminal wealth: either the target wealth is attained or the terminal wealth is zero. We believe that such binary outcomes would be disagreeable to most retirement investors. Indeed, the experiments of Benartzi and Thaler (1999) suggest that investors are highly sensitive to the distribution of terminal wealth.

We are not aware of another paper that considers constraining the terminal wealth to be at most some target capital. Setting a retirement savings wealth goal is in line with advice given to individuals by financial advisers (Greninger et al., 2000). Although it may appear to be rather unambitious to aim at or below a target value rather than above it, we find that there are very appealing consequences. The probability of attaining the target is higher than under the optimal unconstrained strategy. This may be more reassuring to the retirement investor. The quantiles below the target are higher than those for the optimal unconstrained strategy, and they are higher by a constant ratio that can be calculated in advance. In summary, the investor increases their chances of attaining their desired target retirement wealth and, even if they fail to reach it, they still have a higher wealth than if they had no such target.

This paper is organized as follows. Section 2 presents the market model and the investor generic savings behaviour with deterministic cash flows. Our setting can be generalized, but we do not consider random cash flows for simplicity. Section 3 provides the solution to the unconstrained case, where terminal wealth is not bounded. The constrained optimal strategy is shown in Section 4. Section 5 discusses the choice of a target level in the terminal wealth distribution. A numerical illustration and a discussion conclude the paper.

2. Notation and model assumptions

2.1. Market model

We assume investment in a continuous-time market model over a finite time horizon $[0, T]$ for an integer $T > 0$. We refer to T as the *terminal time*.

The market consists of one risky stock and one risk-free bond. The price of the stock is driven by an 1-dimensional, standard Brownian motion $W = \{W(t); t \in [0, T]\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The risk-free bond has price process $\{S_0(t); t \in [0, T]\}$ and the risky stock has price process $\{S_1(t); t \in [0, T]\}$ with dynamics

$$\mathrm{d}S_0(t) = rS_0(t)\,\mathrm{d}t, \qquad \mathrm{d}S_1(t) = S_1(t)\left(\mu dt + \sigma dW(t)\right), \tag{2.1}$$

with $\sigma > 0, S_0(0) = 1$, a.s. and $S_1(0) = 1$, a.s. We assume that $\mu > r \geq 0$. Define the constant market price of risk $\theta := (\mu - r)/\sigma$. The information available to investors is represented by the filtration

$$\mathcal{F}_t := \sigma\{W(s), s \in [0, t]\} \vee \mathcal{N}(\mathbb{P}), \quad \forall t \in [0, T],$$

where $\mathcal{N}(\mathbb{P})$ denotes the collection of all $\mathbb{P}$-null events in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2. Investor

An investor starts with a fixed non-random initial wealth $x_0 > 0$ and plans to make a sequence of known future savings. Define $C(t)$