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Geometric stopping of a random walk and its applications to valuing equity-linked death benefits

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HIGHLIGHTS

- Wiener–Hopf factorization for geometrically stopped random walks is derived.
- Curtate-future-lifetime is approximated by combinations of geometric distributions.
- The logarithm of the stock price process is modeled as a binomial or trinomial tree.
- Closed-form formulas for various equity-linked death benefits are derived.

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ABSTRACT

We study discrete-time models in which death benefits can depend on a stock price index, the logarithm of which is modeled as a random walk. Examples of such benefit payments include put and call options, barrier options, and lookback options. Because the distribution of the curtate-future-lifetime can be approximated by a linear combination of geometric distributions, it suffices to consider curtate-future-lifetimes with a geometric distribution. In binomial and trinomial tree models, closed-form expressions for the expectations of the discounted benefit payment are obtained for a series of options. They are based on results concerning geometric stopping of a random walk, in particular also on a version of the Wiener–Hopf factorization.

1. Introduction

This paper is dedicated to the celebration of the 65th birthday of Professor Rob Kaas. A key motivation for this paper is the problem of valuing Guaranteed Minimum Death Benefits (GMDB) in various equity-indexed and variable annuity contracts. In the literature, the problem is usually studied in a continuous-time setting, e.g., Milevsky and Posner (2001), Ulm (2006, 2008), and Gerber et al. (2012, 2013). In practice, many actuaries use discrete-time models (International Actuarial Association, 2010). One reason for this is that insurance products are monitored periodically. We consider discrete-time models in this paper.

For $t = 0, 1, 2, \ldots$, we model the time-t price of a stock or stock index as

$$S(t) = S(0)a^{X(t)}, \tag{1.1}$$

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where $a > 1$ is a constant and $\{X(t)\}$ is a random walk. If $\{X(t)\}$ is a simple random walk, then (1.1) is the binomial tree model popularized by Cox et al. (1979). In this paper, many explicit results are derived for binomial and trinomial tree models. For ease of discussion, the length of each period is usually assumed to be one year.

We are interested in evaluating expectations of the form

$$\mathrm{E}[v^{K_x+1}b(S(0), S(1), \ldots, S(K_x))], \tag{1.2}$$

where K_x is the curtate-future-lifetime (time until the beginning of the period of death) random variable for a life-age-x, and b is a death benefit function that may depend on the stock-price history up to time K_x. We assume that the death benefit is paid at the end of the period of death.

Let us give some examples. Consider a GMDB rider that guarantees the following death benefit payment,

$$\max(S(K_x), G), \tag{1.3}$$

where G is the guaranteed amount. Because

$$\max(S(K_x), G) = S(K_x) + [G - S(K_x)]_+,$$

the problem of valuing the guarantee becomes the problem of valuing

$$[G - S(K_x)]_+, \tag{1.4}$$

the payoff of a *put* option. Since K_x is a random variable, the put option is of neither the European style nor the American style. It is a *life-contingent* put option (whose valuation in a binomial or trinomial tree model can be obtained by applying formula (7.10) below). Next, suppose that G is not a constant but a fraction, say 90%, of the maximum stock price from time 0 to time K_x. Then the problem is to value the payoff

$$[0.9 \times \max(S(0), S(1), \ldots, S(K_x)) - S(K_x)]_+. \tag{1.5}$$

This path-dependent option is called a *fractional floating strike lookback put* option (which can readily be evaluated by applying formula (9.19)). Our third example arises from the fact that if the stock price rises, a put option such as (1.4) becomes less valuable and the policy may lapse. Thus, instead of (1.4) we may want to consider the payoff

$$I_{(\max(S(0),S(1),\ldots,S(K_x))<H)} \times [G - S(K_x)]_+, \tag{1.6}$$

where $I_{(.)}$ denotes the indicator function and H is a barrier. This is the payoff of an up-and-out put option, a particular form of barrier options that will be studied in Section 8.

To evaluate (1.2), we assume that the random variable K_x and the stock price process $\{S(t)\}$ are independent. As shown in Section 5, the distribution function of K_x can be approximated by linear combinations of geometric distributions. Hence, our problem can be reduced to the evaluation of

$$\mathrm{E}[v^{\tau+1}b(S(0), S(1), \ldots, S(\tau))], \tag{1.7}$$

where τ is an arbitrary geometric random variable independent of $\{S(t)\}$. Also, (1.7) can be factorized as

$$\mathrm{E}[v^{\tau+1}]\tilde{\mathrm{E}}[b(S(0), S(1), \ldots, S(\tau))], \tag{1.8}$$

where tilde signifies that the parameter value of the geometric random variable τ is altered; see (5.5). Thus, our problem can be further reduced to the problem of evaluating

$$\mathrm{E}[b(S(0), S(1), \ldots, S(\tau))], \tag{1.9}$$

for arbitrary geometric random variables τ independent of the stock price process $\{S(t)\}$. For binomial and trinomial tree models, we have derived explicit expressions for (1.9) for b being payoff functions of call options, put options, barrier options, and lookback options.

Geometric stopping of a random walk is the discrete counterpart of exponential stopping of a Lévy process. Sections 2–4 provide a self-contained exposition of results, which are the tools for the subsequent sections.

Let $M(\tau)$ denote the running maximum of $\{X(t)\}$ up to time τ. We show that the random variables $M(\tau)$ and $[X(\tau) - M(\tau)]$ are independent. Hence, under the assumption that the random walk $\{X(t)\}$ is integer-valued, for integers h and j with $h \geq \max(0, j)$,

$$\begin{aligned}&\Pr\{X(\tau) = j, M(\tau) = h\}\\&\quad = \Pr\{M(\tau) = h\}\Pr\{X(\tau) - M(\tau) = j - h\}.\end{aligned} \tag{1.10}$$

To determine the two probabilities on the right-hand side (RHS) of (1.10), we find their probability generating functions by means of the identity

$$\mathrm{E}[z^{X(\tau)}] = \mathrm{E}[z^{M(\tau)}] \times \mathrm{E}[z^{X(\tau)-M(\tau)}]. \tag{1.11}$$

Details of this important step are given in Section 3. Similarly, we can find the joint probability

$$\Pr\{X(\tau) = j, m(\tau) = h\}, \quad h \leq \min(0, j),$$

where $m(\tau)$ denotes the running minimum of $\{X(t)\}$ up to time τ. These joint probabilities are useful for valuing barrier options; see Section 8.

Many contracts have a finite expiry date. This problem may be handled by means of the time-honored actuarial method of Esscher transforms; see Section 10.

The Appendix gives two identities in the trinomial tree case for the joint probability of $X(T) = j$ and $M(T) \geq k$, where T, j and k are integers, with T and k nonnegative and $k \geq j$. In the binomial tree case, these two identities can be found in Föllmer and Schied (2011).

In deriving our formulas, we do not make any assumption whether the expectation (1.2) is calculated with respect to a risk-neutral probability measure. With the payoff being a function of K_x, we are in an incomplete market situation where there is no unique choice of probability measure for valuation.

We should emphasize that results in this paper are not restricted to valuing death benefits. Instead of a time-until-death random variable, we can consider a time-until-catastrophe random variable, and so on. A key assumption is that such a random variable is independent of the stock-price process $\{S(t)\}$.

There is a relatively large literature about equity-linked annuities. Excellent literature reviews can be found in Azimzadeh et al. (2014), Bacinello et al. (2011), and MacKay (2014).

2. Geometric stopping of a random walk

We consider a *random walk* with initial position $X(0) = 0$ and independent and identically distributed (i.i.d.) increments $X_1, X_2, \ldots$. For $t = 1, 2, \ldots$, the position after t steps is

$$X(t) = X_1 + \cdots + X_t. \tag{2.1}$$

Let τ be a geometric random variable (r.v.) independent of the random walk, with

$$\Pr\{\tau = t\} = (1 - \pi)\pi^t, \quad t = 0, 1, 2, \ldots. \tag{2.2}$$

Its *probability generating function* (pgf) is

$$P_\tau(z) = \sum_{t=0}^{\infty} \Pr\{\tau = t\}z^t = \frac{1 - \pi}{1 - \pi z}. \tag{2.3}$$