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Maxentropic approach to decompound aggregate risk losses

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ABSTRACT

A risk manager may be faced with the following problem: she/he has obtained loss data collected during a year, but the data only contains the total number of events and the total loss for that year. She/he suspects that there are different sources of risk, each occurring with a different frequency, and wants to identify the frequency with which each type of event occurs and if possible, the individual losses at each risk event.

The purpose of this methodological note is to examine a combination of disentangling and decompounding procedures, to get as close as possible to that goal. The disentangling procedure is actually a two step process: First, a preliminary analysis is carried out to determine the number of risks groups present. Once that is decided, the underlying model for the frequency of each type of risk is worked out. After that we use the maxentropic techniques in the decompounding stage to determine the distribution of individual losses that aggregated yield the observed total loss.

1. Introduction

An interesting directive proposed in the Basel II agreement is the possibility for the banks to develop their own advanced measurement approach (AMA) to compute the regulatory capital based on the loss distribution approach (LDA). This approach, modeled on that developed by the insurance industry and later included within the Basel II in the advanced measurement approach (AMA) for operational risk, is used to determine the distribution of the total loss from two complementary ingredients, to wit, the frequency and the severity of the individual losses measured (or observed).

Let N_h denote the number losses of certain type labeled by $h = 1, \ldots, H$, occurring in a given time interval, H being the total number of risk sources, and $X_{h,k}$, $k \geq 1$ a sequence of positive real-valued random variables that represent the size of the kth loss of type h. With this notation, the quantity of interest is the compound model given by

$$S = \sum_{h=1}^{H} S_h \quad \text{where } S_h = \sum_{k=1}^{N_h} X_{h,k} \text{ with } S_k = 0 \text{ if } N_h = 0. \tag{1}$$

Here, $h = 1, \ldots, H$ labels the business line/type of the institution. In (1), it is supposed for each h that the $X_{h,k}$, $k \geq 1$ are independent and identically distributed and independent of the N_h, and we shall denote by X_h a random variable having that common distribution.

An interesting problem confronting the risk manager occurs at the level of the S_h. It may be the case that the frequency of events of type h is actually the sum of at least two different types of events, i.e., it may happen that $N_h = N_h^1 + N_h^2$ during the observation period, even though the individual losses in each case may be the same. In this case, the risk analyst may want to know how many types of risk events of each type occur, and if possible, what is the distribution of individual losses. This is of interest because it is at that level where loss prevention or mitigation may be applied. This question motivates the problem that we shall examine here by a combination of procedures: one (called disentangling) which consists of determining how many risk sources are there and their frequencies, and the other, the decompounding, is used to determine what is the statistical nature of the individual severities. Both procedures will be explained in Section 2, in which we establish the methodological aspects of the paper. We shall see that unless we know that the distributions of individual losses are the same, all that one is able to obtain is a mixture of distributions, a kind of collective individual loss distribution, which becomes the true individual distribution only when it is known that all are the same.

The disentangling procedure was proposed in Gomes-Gonçalves and Gzyl (2014), and we briefly review it here for the sake of completeness. The decompounding problem was treated for the first time by Buchmann and Grübel (2004). There they estimate the

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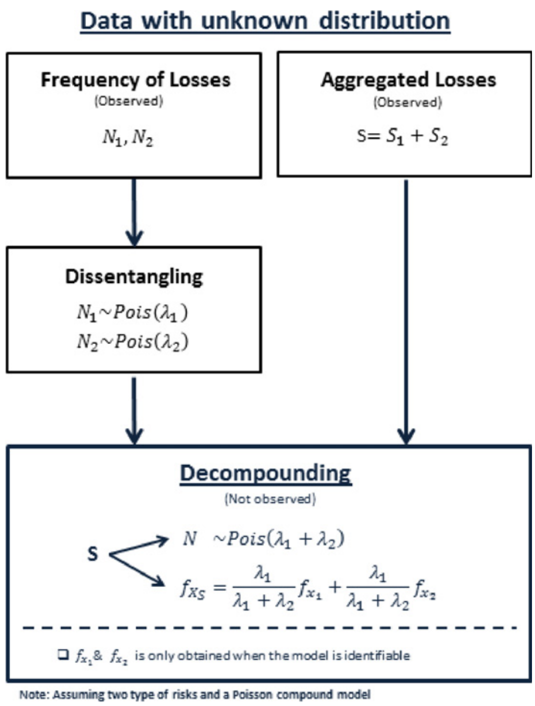

Fig. 1. Procedure.

frequency rate parameter and the individual severity distribution, in a univariate compound Poisson distribution non-parametrically through a discrete Panjer inversion. Later on, a different approach to the problem was presented in Van Es et al. (2007), who proposed a kernel type, nonparametric density estimator to obtain the individual losses of a univariate and bivariate compound Poisson distribution. They report good results in the numerical examples presented. Bøgsted and Pitts (2010) extended the work of Buchmann and Grübel (2004), and constructed a generalization of a nonparametric estimator for any compound base process given any parametric model for the distribution of the frequencies. We shall carry out the decompounding procedure using the maximum entropy method, for which the input shall be the Laplace transform of the aggregated loss S.

The remainder of the paper is organized as follows: the second section is devoted to a more detailed description of the methodology, the third to a review of the maximum entropy method, and the fourth to the discussion of a few numerical examples, after which we gather some concluding remarks.

2. Methodological preliminaries

In this section we lay down the basic framework in which we shall be working. We refer the reader to Fig. 1 for graphic display of the setup. As mentioned, the number of risk events during a certain period of time is recorded as well as the total loss during the period. It is known, or suspected, that there may be more of one source of risk present in the recorded aggregated loss, thus the first issue to take care of, is to determine the number of different risk types and their statistical nature. Once this is achieved, the next order of business is to determine the distribution of the individual risks.

In actual practice, before applying the disentangling procedure to determine the statistical nature of the various frequencies of events, a first step consisting of determining how many different type H of events are there will be applied. Once that number is known ($H = 2$ in the diagram), we apply the disentangling procedure described below. The output of this preprocessing stage is the specification of $H = 2$ integer valued random variables. If we suppose that the individual losses of each type are the same, invoking (1), it is clear for the independence assumptions that the Laplace transform $\psi(\alpha)$ of the total loss is given by

$$\begin{aligned}\psi(\alpha) &= E[e^{-\alpha S}] = E[e^{-\alpha S_1}]E[e^{-\alpha S_2}]\\ &= G_{N_1}(\phi(\alpha))G_{N_2}(\phi(\alpha))\end{aligned} \tag{2}$$

where, of course, $\phi(\alpha) = E[e^{-\alpha X}]$ denotes the Laplace transform of a random variable having the common distribution of the individual losses in (1). Thus, once that we have determined the distributions of the N_i, $i = 1, 2$, and computed $\psi(\alpha)$ from the data, we can solve (2) for $\phi(\alpha)$ and use it as input in the maxentropic methods to obtain the probability density of the individual losses from it. This is the second stage of the process.

The result of the methodology summed up in 1 is a mixture of discrete distributions to describe the total frequency N of events, that will be used as input of our second step, the decompounding methodology to determine the distribution of individual loses present in the aggregate loss.

To exemplify, suppose that the frequencies are Poisson(λ_i), for $i = 1, \dots, H$, and that the individual severities are distributed according to f_{X_i}. In this case, a simple computation shows that in this case (2) becomes

$$\psi(\alpha) = e^{-\sum_{h=1}^{H}\lambda_i(E[e^{-\alpha X_h}]-1)} = e^{-\lambda(E[e^{-\alpha\hat{X}}]-1)} \tag{3}$$

where we put $\lambda = \sum_{h=1}^{H}\lambda_i$ and $\hat{X}$ is a random variable whose density is the mixture $f_{\hat{X}} = \sum_{h=1}^{H}\frac{\lambda_i}{\lambda}f_{X_h}$. That is, the aggregated risk is the result of compounding a risk produced with a Poisson intensity equal to the sum of the individual intensities and individual loss with probability density equal to a weighted average of the individual densities, with respect to weights equal to the proportion of the individual frequency relative to the total frequency.

Following Wang (1997) we may extend the previous result as follows.

Proposition 1. *Suppose that the H compound risks to be aggregated have frequencies N_i with a common mixing distributions $F(\theta)$ such that, given θ the $N_{|\theta} \sim P(\theta\lambda_i)$ and are independent. Let $S = \sum_{h=1}^{H} S_i$ be as above. Then $S \sim \sum_{n=1}^{N}\hat{X}_n$ with $N_{|\theta} \sim P(\theta\lambda)$, where $\lambda = \sum_{h=1}^{H}\lambda_i$, and as above, the $\hat{X}_n$ have common density $\sum_{h=1}^{H}\frac{\lambda_I}{\lambda}f_{X_h}$.*

Proof. The proof is actually easy, and hinges on the fact that $\frac{\lambda_i}{\lambda} = \frac{\theta\lambda_i}{\theta\lambda}$ is independent of the mixing parameter θ, thus proceeding as if to prove (3) we would be led to

$$\psi(\alpha) = \int e^{-\theta\lambda(E[e^{-\alpha\hat{X}}]-1)}dF(\theta) = E\left[e^{-\alpha\sum_{n=1}^{N}\hat{X}_n}\right]$$

as claimed □

Notice that for the proposed mixing we have $\lambda_i/\sum_j\lambda_j = E[N_i]/\sum jE[N_j]$. This type of mixing appears as well in what Wang (1997) calls common shock models, which include correlated Poisson frequencies. The class of frequencies for which this is valid includes the Poisson and Negative Binomial random variables. These results were turned into a proposal to define an equivalent individual loss distribution by Wang, which we state as follows.

Definition 1 (*Equivalent Mixture*)**.** Let f_{X_h} denote the common probability density of the individual losses for the compound losses of the hth type, $h = 1, \dots, H$, and let N_h denote the respective loss frequency. The aggregated individual loss has a density given by

$$f_{\hat{X}} = \sum_{h=1}^{H}\frac{E[N_h]}{E[N_{agg}]}f_{X_h}, \tag{4}$$