Insurance: Mathematics and Economics 64 (2015) 409–416

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Convex ordering for insurance preferences

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ARTICLE INFO

Article history:
Received April 2015
Received in revised form
June 2015
Accepted 5 June 2015
Available online 31 July 2015

Keywords:
Convex ordering
Karlin–Novikoff–Stoyan–Taylor crossing conditions
Value-at-Risk
Average Value-at-Risk
Optimal insurance decision problem

ABSTRACT

In this article, we study two broad classes of convex order related optimal insurance decision problems, in which the objective function or the premium valuation is a general functional of the expectation, Value-at-Risk and Average Value-at-Risk of the loss variables. These two classes of problems include many existing and contemporary optimal insurance problems as interesting examples being prevalent in the literature. To solve these problems, we apply the Karlin–Novikoff–Stoyan–Taylor multiple-crossing conditions, which is a useful sufficient criterion in the theory of convex ordering, to replace an arbitrary insurance indemnity by a more favorable one in convex order sense. The convex ordering established provides a unifying approach to solve the special cases of the problem classes. We show that the optimal indemnities for these problems in general take the double layer form.

1. Introduction

Optimal insurance decision problem has long been one of the most popular research topics in the insurance context due to its immediate practical consequence. The optimality of the deductible contracts for policyholders subject to the classical expected value premium principle was first proven by Borch (1960) for the minimization of the variance of the retained loss, and by Arrow (1974) for the maximization of the expected utility (EU) of the terminal wealth. Since then, intense effort has been observed in the literature to solve similar problems under various model settings with different objective functions as well as imposing various constraints that lead to a variety of optimality results. For example, see Asimit et al. (2013a,b), Balbás et al. (2009), Bernard and Tian (2009), Cai et al. (2008), Centeno and Guerra (2010), Cheung et al. (2013, 2014, 2015), Kaluszka and Okolewski (2008), Sung et al. (2011), and Tan et al. (2011), and the references therein.

The notion of *stochastic ordering*, in particular *convex ordering*, have been well developed and they are essential for comparing risky alternatives in decision analysis based on different criteria. For example, convex ordering arranges risks by their variations with respect to the value of same means, and consequently allows the decision maker to choose the "least risky" alternative. Convex ordering has been thoroughly applied for solving various problems in economics, finance and actuarial science, which demonstrates its usefulness and importance. For instance, it can be applied to compare the aggregate risk of a portfolio, in which the comonotonicity structure among the risks attains the upper bound of the convex order. For comprehensive studies and other applications in convex ordering, see Denuit et al. (2005), Denuit and Dhaene (2012), Dhaene et al. (2002, 2006, 2012), Kaas et al. (1994, 2008), Müller and Stoyan (2002), Rüschendorf (2013), and Shaked and Shanthikumar (2007), and the references therein.

The convex ordering approach to solve the optimal insurance decision problem was first adopted by Ohlin (1969) of minimizing a measure of the dispersion of the retained and ceded losses. The crucial mathematical tool employed by Ohlin (1969) is the 'Karlin–Novikoff once-crossing criterion' by Karlin and Novikoff (1963) for (increasing) convex ordering. Later, Gollier and Schlesinger (1996) used the same approach to extend the result of Arrow (1974) through maximizing an increasing convex order preserving objective functional of the terminal wealth. More recently, this approach was re-exploited to solve various optimal insurance decision problems. For instance, Cai and Wei (2012) solved the multivariate risk minimizing problems in which the risks are positively dependent. Chi and Tan (2013) considered the optimal insurance problems under which the premium principle is a certain convex order preserving functional.

In this paper, we study two broad classes of convex order related optimal insurance decision problems:

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http://dx.doi.org/10.1016/j.insmatheco.2015.06.005

(I) maximizing a concave order preserving functional of the terminal wealth of the insured with the premium principle specified by a general function of the expectation, Value-at-Risk (V@R), and Average Value-at-Risk (AV@R) of the indemnity; and,
(II) minimizing another general function of expectation, V@R and AV@R of the terminal loss of the insured with the premium valuated by a general function of the expectation and a convex order preserving functional of the ceded loss.

Both classes include many existing and contemporary optimal insurance problems as interesting examples as we shall show in later sections.

Since the problem settings involve the convex order preserving functionals, it is natural to apply the convex ordering approach on solving for these two problem classes. Instead of using the 'Karlin–Novikoff once-crossing criterion' by Karlin and Novikoff (1963), we adopt the '*Karlin–Novikoff–Stoyan–Taylor crossing conditions*', developed by Stoyan (1983) and Taylor (1983) and named by Hürlimann (1998, 2008a,b), which is a generalization of the once-crossing condition. By exploiting this multiple-crossing criterion, we are able to

(i) rank the insurance indemnities in terms of their convex orders together with a greater flexibility than that through the once-crossing condition; and,
(ii) provide a unifying approach to solve for two classes of optimal insurance decision problems (I) and (II) by using the convex ordering obtained in (i).

The organization of our paper is as follows. In Section 2, two classes of optimal insurance problems with the corresponding optimality criterion and constraint are formulated. The main theorem using the multiple-crossing conditions to establish the convex ordering of the insurance indemnities are presented in Section 3. Resolutions of the special cases of two classes of problems formulated in Section 2 are illustrated as the corollaries in Sections 4 and 5.

2. Preliminaries and problem formulation

2.1. Preliminaries

We first recall the definitions and results of several stochastic orderings. For a comprehensive review of the theory and applications, see the references in the first paragraph in Introduction. In this section, Y and Z are random variables with cumulative distribution functions F_Y and F_Z.

Definition 2.1. Y is said to be smaller than Z in the convex (concave, increasing convex, increasing concave, resp.) order if for all convex (concave, increasing convex, increasing concave, resp.) functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[\varphi(Y)] \le \mathbb{E}[\varphi(Z)]$, provided that the expectations exist. The convex (concave, increasing convex, increasing concave, resp.) ordering is denoted as $Y \le_{cx} Z$ ($Y \le_{cv} Z$, $Y \le_{icx} Z$, $Y \le_{icv} Z$, resp.).

Since $Y \le_{cx} Z$ is equivalent to $Z \le_{cv} Y$, and $Y \le_{icx} Z$ is equivalent to $-Y \ge_{icv} -Z$, and we shall only make use of the results in convex and increasing convex order in this article, we only present the following summary of useful results for the convex and increasing convex order. The counterpart results for the concave and increasing concave order are similar. In what follows, all stated moments are assumed to be finite.

Proposition 2.1.

(i) *If $Y \le_{cx} Z$, then $\mathbb{E}[Y] = \mathbb{E}[Z]$ and $Var(Y) \le Var(Z)$. Also, if $Y \le_{icx} Z$, then $\mathbb{E}[Y] \le \mathbb{E}[Z]$.*
(ii) *Define $\pi_Y(t) \triangleq \mathbb{E}[(Y-t)_+]$ as the stop-loss transform of Y. Then, $Y \le_{icx} Z$ if, and only if, $\pi_Y(t) \le \pi_Z(t)$ for any real numbers t. Furthermore, if $\mathbb{E}[Y] = \mathbb{E}[Z]$, then $Y \le_{cx} Z$ if, and only if, $\pi_Y(t) \le \pi_Z(t)$.*

Notice that the results of convex order and increasing convex order are analogous to each other; indeed, we have the following equivalence of these two orders provided that the means of Y and Z are equal.

Proposition 2.2. *$Y \le_{cx} Z$ if, and only if, $Y \le_{icx} Z$ and $\mathbb{E}[Y] = \mathbb{E}[Z]$.*

To facilitate further use of convex and increasing convex orderings, Karlin and Novikoff (1963) provided sufficient conditions in terms of the cumulative distribution functions, known as 'Karlin–Novikoff once-crossing criterion'.

Definition 2.2. The distribution functions F_Y and F_Z are said to be crossing $r \ge 1$ times if there exist

$$\xi_{0,2} < \xi_{1,1} \le \xi_{1,2} < \xi_{2,1} \le \xi_{2,2} < \cdots < \xi_{r,1} \le \xi_{r,2} < \xi_{r+1,1},$$

where $\xi_{0,2} \triangleq \inf\{x : F_Y(x) \ne F_Z(x)\}$ and $\xi_{r+1,1} \triangleq \sup\{x : F_Y(x) \ne F_Z(x)\}$, such that, for each $i = 1, 2, \ldots, r$,

(i) for any $x \in (\xi_{i-1,2}, \xi_{i,1})$ and $y \in (\xi_{i,2}, \xi_{i+1,1})$,

$$(F_Y(x) - F_Z(x))(F_Y(y) - F_Z(y)) < 0; \quad \text{and}$$

(ii) if $\xi_{i,1} < \xi_{i,2}$, $F_Y(z) = F_Z(z)$ for any $\xi_{i,1} \le z < \xi_{i,2}$.

Theorem 2.3. *Assume that $\mathbb{E}[Y] = \mathbb{E}[Z]$ (resp. $\mathbb{E}[Y] \le \mathbb{E}[Z]$). If F_Y and F_Z cross once, and $F_Y(x) - F_Z(x) < 0$ for $\xi_{0,2} < x < \xi_{1,1}$, then $Y \le_{cx} Z$ (resp. $Y \le_{icx} Z$).*

In addition, in this paper we shall make use of the following generalization by Stoyan (1983) and Taylor (1983), coined as 'Karlin–Novikoff–Stoyan–Taylor crossing conditions' by Hürlimann (1998, 2008a,b).

Theorem 2.4. *Assume that F_Y and F_Z cross $n \ge 1$ times. Then $Y \le_{icx} Z$ if, and only if, one of the following two cases is satisfied:*

<u>*Case 1*</u>

(i) *There is an even number of crossings $n = 2m$ for some $m = 1, 2, \ldots$;*
(ii) *$F_Y(x) - F_Z(x) > 0$ for $\xi_{0,2} < x < \xi_{1,1}$; and*
(iii) *for any $j = 1, 2, \ldots, m$, $\pi_Y(\xi_{2j-1,2}) \le \pi_Z(\xi_{2j-1,2})$.*

<u>*Case 2*</u>

(i) *$\mathbb{E}[Y] \le \mathbb{E}[Z]$;*
(ii) *there is an odd number of crossings $n = 2m - 1$ where $m = 1, 2, \ldots$;*
(iii) *$F_Y(x) - F_Z(x) < 0$ for $\xi_{0,2} < x < \xi_{1,1}$; and*
(iv) *if $m \ge 2$, for any $j = 1, 2, \ldots, m-1$, $\pi_Y(\xi_{2j,2}) \le \pi_Z(\xi_{2j,2})$.*

Applying Proposition 2.2 yields an analogous theorem for the convex order by noting that when $\mathbb{E}[Y] = \mathbb{E}[Z]$, Case 1 in Theorem 2.4 cannot be valid as before.

Theorem 2.5. *Assume that F_Y and F_Z cross $n \ge 1$ times. Then $Y \le_{cx} Z$ if, and only if,*

(i) *$\mathbb{E}[Y] = \mathbb{E}[Z]$;*
(ii) *there is an odd number of crossings $n = 2m - 1$ where $m = 1, 2, \ldots$;*
(iii) *$F_Y(x) - F_Z(x) < 0$ for $\xi_{0,2} < x < \xi_{1,1}$; and*
(iv) *if $m \ge 2$, for any $j = 1, 2, \ldots, m-1$, $\pi_Y(\xi_{2j,2}) \le \pi_Z(\xi_{2j,2})$.*