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Entrance times of random walks: With applications to pension fund modeling

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ABSTRACT

The purpose of the paper is twofold. First, we consider entrance times of random walks, i.e. the time of first entry to the negative axis. Partition sum formulas are given for entrance time probabilities, the nth derivative of the generating function, and the nth falling factorial entrance time moment. Similar formulas for the characteristic function of the position of the random walk both conditioned on entry and conditioned on no entry are also established. Second, we consider a model for a with-profits collective pension fund. The model has previously been studied by approximate methods, but we give here an essentially complete theoretical description of the model based on the entrance time results. We also conduct a mean–variance analysis for a fund in stationarity. To facilitate the analysis we devise a simple and effective exact simulation algorithm for sampling from the stationary distribution of a regenerative Markov chain.

1. Introduction

This paper analyzes a "with-profits collective pension scheme"; this type of scheme and variants hereof are widespread, among other places, in the Nordic countries and the Netherlands. Members of the scheme are guaranteed a minimum benefit. The guarantees are a liability for the pension fund for which it must reserve an amount of money equal to the net present value of future guaranteed benefits (the *reserve*). In addition to the already guaranteed benefits members may receive *bonuses* in the form of increased guarantees. Bonuses are attributed periodically, e.g. annually, when the ratio of total assets to the reserve (the *funding ratio*) is sufficiently high. The phrase "with-profits" refers to this profit-sharing mechanism.

Assets in excess of the reserve are termed the *bonus potential*. The bonus potential allows the fund to invest in risky assets by absorbing adverse investment results. The scheme is "collective" in the sense that the bonus potential is considered common to all members. It is also collective in the sense that the investment strategy and bonus policy is the same for all members. Collective funds generally benefit from economy of scale in the form of low administration and investment costs. The flip side is the lack of an individual investment strategy.

We consider a model for a collective pension fund in which a bonus is attributed when the funding ratio exceeds a given *bonus threshold*. The fund follows a CPPI (Constant Proportion Portfolio Insurance) investment strategy in order to stay solvent, i.e. to ensure that total assets exceed the reserve. The paper gives an essentially complete description of the fund dynamics including the time between bonuses, the (conditional) expected bonus percentage and the (conditional) expected funding ratio. The analysis is based on a detailed study of an embedded one-sided random walk obtained by a transformation of the funding ratio process sampled at the discrete set of time points where a bonus can be attributed. We consider both a fund starting at the bonus threshold and a fund in stationarity. Furthermore, we use the results to perform a mean–variance analysis of a standardized benefit payout. The analysis is performed for a fund in stationarity representing the "average" member. To facilitate the analysis we also employ an *exact simulation algorithm*, which might be of independent interest.

The theoretical foundation for the analysis is a series of new results for entrance times of random walks derived in this paper. For a random walk started at the origin the entrance time is the time of entry into $(-\infty, 0]$. The main theoretical results are

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partition sum formulas for the nth derivative of the entrance time generating function and for the nth falling factorial entrance time moment. The latter result generalizes the well-known formula for the mean entrance time. We also give a partition sum formula for entrance time probabilities, and a similar formula for the position of the random walk conditioned on entrance at time n. These results are implicit in Spitzer (1956) and Asmussen (2003), but the proofs are new and simpler. Finally, we give a new partition sum formula for the position of the random walk conditioned on entrance taken place after time n.

The results allow explicit calculations of the entrance time probabilities and moments in terms of the marginal distribution of the random walk. The computational effort gradually becomes prohibitive, but the first 100, say, entrance time probabilities and moments are computationally feasible. We use the results to study the one-sided random walk embedded in the funding ratio process by utilizing the fact that a one-sided random walk and its associated (unrestricted) random walk are identical up to the time of first entry into $(-\infty, 0]$. However, the results are generally applicable and not limited to our pension fund application.

Optimizing utility from terminal wealth for an individual saving for retirement is treated by numerous papers. The foundation was laid by Richard (1975) and to mention a few, who among other results obtain optimal investment strategies, there is, Huang and Milevsky (2008) who allow for unspanned labor income; Huang et al. (2008) who separate the breadwinner income process from the family consumption process; Steffensen and Kraft (2008) who generalize to a multi-state Markov chain framework typically used by actuaries for modeling a series of life history events; Bruhn and Steffensen (2011) who generalize to a multi-person household, with focus on a married couple with economically and/or probabilistically dependent members; Kwak et al. (2011) who also consider a household but focus on generation issues; Kronborg and Steffensen (2013) who calculate the optimal investment strategy for a pension saver in the presence of a minimum rate guarantee; and Gerber and Shiu (2000) who present a comprehensive discussion of terminal utility optimization in a pension saving context. There is also a vast literature on modern investment management, founded by Markowitz (1952) and Merton (1971), aimed at finding optimal investment strategies without the pension aspect.

In contrast to the literature cited above this paper takes the point of view of a pension fund where a group of people share a common investment strategy. Investment gains are shared through a bonus strategy by which collective funds above a threshold are transferred to individual guarantees. Based on results on utility optimization of durable goods by Hindi and Huang (1993) it can be shown that the optimal bonus strategy is to continuously attribute bonuses whenever the funding ratio exceeds a certain barrier—thereby not allowing the funding ratio to exceed the barrier. In this paper, and in real life, the transfer is done periodically rather than continuously. References for continuous-time analysis of pension schemes taking both assumed (technical) returns, realized returns and bonus into account include Norberg (1999), Steffensen (2000), Norberg (2001), Steffensen (2004) and Nielsen (2006).

We assume the fund to follow a CPPI strategy. This strategy ensures that the fund remains funded, and it is ''locally'' optimal if we consider the periods between possible bonus attributions as local horizons. More precisely, Preisel et al. (2010) point out that CPPI is optimal in a finite horizon setting with HARA-utility and a subsistence level corresponding to a terminal funding ratio of one. CPPI strategies are treated, for the unrestricted case, by Cox and Huang (1989), and for the restricted case, by Teplá (2001). Using a CPPI strategy, and thereby avoiding insolvency, as done in this paper, stands in contrast to the literature on constructing contracts that are fair between owners and policyholders, see e.g. Briys and de Varenne (1997) and Grosen and Jørgensen (2000).

The current paper is related to Preisel et al. (2010), Kryger (2010) and Kryger (2011). We use the same underlying funding ratio dynamics, but the pension product and the terms by which members enter and leave the fund differ. In the cited papers members pay a fixed share (possibly zero) of contributions to the bonus potential on entry. This raises a number of issues regarding intergenerational fairness. In the present setup the share depends on the funding status of the fund in such a way that the contract is always financially fair. Methodologically, the cited papers use various analytical approximations while the current paper relies almost exclusively on exact results.

The main insight of Preisel et al. (2010) is that a given year's apparent success of a large bonus resulting from a high equity allocation can come at the even higher price of subsequent large losses trapping the company at a low funding ratio for a long period. They also derive approximations to the expected bonus and funding ratio in stationarity. Kryger (2010) finds optimal investment strategies for power utility and mean–variance criteria. For fixed values of the bonus threshold, he finds optimal investment strategies in the class of CPPI strategies for a fund in stationary. It is found that different investment strategies imply only modest differences in utility and, hence, that an investment collective can accommodate quite different attitudes towards risk. Finally, Kryger (2011) studies the impact of the pension design on efficiency and intergenerational fairness.

The rest of the paper is organized as follows. Section 2 presents the theoretical contributions on entrance times and moments of random walks. Section 3 describes the pension fund model, and Section 4 applies the random walk results to study bonus waiting times, the bonus size and the funding ratio. Results are given for a fund started at the bonus threshold and for a fund in stationarity. Section 5 contains a comprehensive application including a mean–variance analysis. It also explains the exact simulation algorithm used in the analysis. Finally, the Appendix contains proofs for the results of Section 2 and additional lemmas.

2. Random walks

In this section we present a series of results on entrance times and conditional characteristic functions of random walks. The results will be used in subsequent sections to provide a detailed description of the distribution of bonus times, bonus size and the funding ratio of the collective pension fund model under study. However, the results are generally applicable and can be applied in many other contexts as well.

The entrance time of a random walk is defined as the (first) time of entry into $(-\infty, 0]$ after time 0. The results to follow devise how a number of quantities related to entrance times can be computed as sums over partition sets. We present both new results and existing results with new and simpler proofs. The results fall in three parts.

First, we derive a closed-form formula for the entrance time probabilities of a random walk started at the origin (Theorem 2.3). This result is also implicit in the seminal paper by Spitzer (1956), but we give here a simpler self-contained proof. Second, we derive an expression for the nth derivative of the generating function for the entrance time (Theorem 2.5), which we subsequently use to derive a formula for the factorial moments (Theorem 2.6). These results are new. Third, we derive formulas for the characteristic function of the position of the random walk conditioned on entrance at time n (Theorem 2.8) and on entrance after time n (Theorem 2.9). The first of these results is known, but the proof is new, while the second result is new. Most of the proofs rely on combinatorial arguments, some of which might be of independent interest, in particular Lemma A.1.