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Marginal Indemnification Function formulation for optimal reinsurance

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ABSTRACT

In this paper, we propose to combine the Marginal Indemnification Function (MIF) formulation and the Lagrangian dual method to solve optimal reinsurance model with distortion risk measure and distortion reinsurance premium principle. The MIF method exploits the absolute continuity of admissible indemnification functions and formulates optimal reinsurance model into a functional linear programming of determining an optimal measurable function valued over a bounded interval. The MIF method was recently introduced to analyze the reinsurance model but without premium budget constraint. In this paper, a Lagrangian dual method is applied to combine with MIF to solve for optimal reinsurance solutions under premium budget constraint. Compared with the existing literature, the proposed integrated MIF-based Lagrangian dual method provides a more technically convenient and transparent solution to the optimal reinsurance design. To demonstrate the practicality of the proposed method, analytical solution is derived on a particular reinsurance model that involves minimizing Conditional Value at Risk (a special case of distortion function) and with the reinsurance premium being determined by the inverse-S shaped distortion principle.

1. Introduction

Reinsurance, if exploited appropriately, can be an effective risk management tool for an insurer. In a typical reinsurance contract, the insurer pays a certain amount of premium to a reinsurer in return for some indemnification when losses occur from a designated risk. This indemnification is always a function of the risk, and the premium is determined by the resulting indemnification function, together with a given premium principle. While a higher stipulated indemnification implies a lower risk exposure to the insurer, this is achieved at the expense of a higher upfront reinsurance premium. This demonstrates the classical tradeoff between risk retention and risk transfer and the problem of optimal reinsurance is to address the optimal risk sharing between insurer and reinsurance for a given prescribed objective and constraints.

Two pioneering works on optimal reinsurance are attributed to Borch (1960) and Arrow (1963). Borch (1960) demonstrated that the stop-loss reinsurance is the best contract if the insurer measures risks by variance and the reinsurer prices risks by the expected value premium principle. Arrow (1963) also showed that the stop-loss reinsurance is an optimal one if the insurer is an expected utility maximizer under the assumption of the expected value premium principle. These fundamental results have been generalized in a number of interesting and important directions. Just to name a few, Kaluszka (2001) extended the Borch's result by considering the mean–variance premium principle, while Young (1999) elaborated Arrow's result by taking Wang's premium principle into account.

More recently, there is a surge of interest in formulating the optimal reinsurance problem involving more sophistical risk measures such as Value at Risk (VaR), Conditional Value at Risk (CVaR) and more generally distortion risk measures. For example, Cai and Tan (2007), Cai et al. (2008), Cheung (2010) and Tan et al. (2011) discussed the minimization of VaR and CVaR of the insurer's total risk exposure with expected value premium principle. Cheung et al. (2014) further explored Tan et al. (2011)'s results under the general law-invariant convex risk measure. Balbás et al. (2009) also studied the optimal reinsurance problem when risk is measured by a general risk measure. Chi and Tan (2013),

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and Chi and Weng (2013) considered VaR and CVaR with premium principles which preserve the convex ordering. Zheng and Cui (2014) designed the optimal reinsurance contract under distortion risk measure, but assuming that the distortion function is piecewise concave or convex. Cui et al. (2013) studied a general model involving the distortion risk measure and the distortion premium principle. Cheung and Lo (in press) extended the model of Cui et al. (2013) to a cost–benefit framework. Assa (2015) demonstrated that the optimal reinsurance model of Cui et al. (2013), without the premium constraint, can be tackled more elegantly via a marginal indemnification function (MIF) formulation.

The primary objective of the present paper is to extend the MIF-based method so as it can be used to derive analytically the solution to the distortion risk measure based reinsurance model in the presence of a premium budget constraint. It is well-known that in many optimization problems, the complexity of the optimization problem can be significantly increased by merely imposing a constraint. In particular, one often discovers that while an optimization procedure can be used to solved an unconstrained optimization problem analytically, the same procedure may no long be applicable when a constraint is imposed on the model. This is precisely the issue with the method of MIF proposed by Assa (2015). As demonstrated in Assa (2015) that without the premium constraint the MIF formulation elegantly solves the reinsurance model of Cui et al. (2013). The same method, however, cannot be readily used in the presence of the premium budget.

The MIF method makes full use of the absolute continuity of admissible ceded loss functions. It is well known that an absolute continuous function over real line is, out of a Lebesgue null set, differentiable. The derivative of the ceded loss function is called marginal indemnification function, because it measures the increase in ceded loss per unit of increase in the group-up loss. It should be pointed out that Balbás et al. (2015) have independently proposed the MIF formulation for optimal reinsurance models, though the term "MIF" was not used. The authors considered a general mean-risk reinsurance model under uncertainty of the group-up risk and formulated the reinsurance model with the derivative of the retained loss function being the decision variable, which they referred to as "sensitivity". Moreover, they proposed to impose a lower bound on the decision variable to effectively eliminate the moral hazard from the insurer. This translates to an upper bound on the MIF in our formulation. For $\widetilde{\mathbb{E}}_\rho$-translation invariant risk measures (which satisfy subadditivity), Balbás et al. (2015) developed two duality methods of transforming optimal reinsurance models, which may be non-linear, into functional linear programming problems.

The objective of this paper is to demonstrate that by integrating MIF with a Lagrangian method, one can derive explicit optimal reinsurance policies for problems with a budget constraint on reinsurance premium and bounds on the derivative of admissible ceded loss functions. Compared to the approach of Cui et al. (2013) for solving the same reinsurance model with premium budget constraint, our proposed integrated MIF and Lagrangian method possesses at least the following three advantages. Firstly, it is simpler and more transparent. More specifically, the approach of Cui et al. (2013) critically depends on a pre-conjectured candidate solution. This implies we need to first guess an optimal solution and then apply certain comparison analysis to prove its optimality. Their method, therefore, requires us to have a preconception on the shape of the optimal solution in order to justify its optimality. Our integrated method, on the other hand, does not require any preanalysis on the shape of optimal solutions. Secondly, due to the nature of the procedure in searching for the solution developed by Cui et al. (2013), it is difficult to discuss the uniqueness of optimal solution. In contrast, the uniqueness of solution can be easily studied, and the non-uniqueness of solutions can also be uncovered from our optimization procedure. Thirdly, even if bounds are imposed on the derivative of the admissible ceded loss functions, our proposed integrated method can similarly be used to derive the explicit solutions of the models.

To highlight the practicality of our proposed solution, we consider a particular reinsurance model that minimizes CVaR (a special distortion risk measure) and with the premium being dictated by the inverse-S shaped distortion (ISSD) premium principle. The ISSD premium principle is a distortion premium principle with a distortion function such that it has derivative which changes from being strictly decreasing to being strictly increasing derivative at a certain point. Thus, it encompasses both the concave and convex distortion premium principles as special cases. Indeed, as it will become clear in Section 5, the optimal solutions for either a concave or a convex distortion premium principle can be recovered from those we obtained for the ISSD premium principle as special cases.

Another important feature of the ISSD premium principle is its economic interpretation in that the insurance provider may overweight not only large losses but also small ones in underwriting the insurance risks. This is consistent with the empirically observed phenomena in psychological experiments (Quiggin, 1982, 1992; Tversky and Kahneman, 1992; Tversky and Fox, 1995; Gonzalez and Wu, 1999). Furthermore, Kaluszka and Krzeszowiec (2012) introduced a premium principle from the perspective of the Cumulative Prospect Theory (CPT). The ISSD premium principle can also be viewed as a special CPT premium principle corresponding to a linear utility function and a zero reference point. Unlike the concave distortion premium principle, the ISSD premium principle has not received much attention in the actuarial literature. This, in part, can be attributed to its relatively new concept and its short history. Other reasons could be due to the possibility that the ISSD introduces additional technical hurdles, such as non-convex order property, for solving optimization problems.

The rest of the paper proceeds as follows. In Section 2, we formally specify our optimal reinsurance models and develop their corresponding MIF's. Section 3 gives the optimal solutions for the model without premium budget constraint. In Section 4, we integrate the Lagrangian dual method with the MIF formulation and derive explicit solutions for the model with reinsurance premium budget constraint. In Section 5, we demonstrate the practicality of our proposed approach by resorting to a specialized example involving risk measure CVaR and reinsurance premium principle ISSD. Section 6 concludes the paper.

2. Model setup

Throughout the paper, all the random variables are defined on a common probability space $(\Omega, F, \mathbb{P})$. The indicator function is denoted by $\mathbf{1}_A(s)$, i.e., $\mathbf{1}_A(s) = 1$ for $s \in A$ and $\mathbf{1}_A(s) = 0$ for $s \notin A$. The capital letter X is exclusively used to denote the non-negative random variable for which the insurer seeks reinsurance coverage and $M \triangleq \text{esssup}X$. For convenience, the domain of the random variable X is consistently denoted by $[0, M]$. While this suggests that the domain of X is a bounded interval, it should be emphasized that all the results obtained in the paper hold even if $\text{esssup}X = \infty$; i.e. even if $[0, M]$ is replaced by $[0, \infty)$.

The problem of optimal reinsurance is concerned with the optimal partitioning of X into $f(X)$ and $r(X)$ such that $X = f(X) + r(X)$, where f and r are two measurable functions defined over $[0, M]$. Here $f(X)$ represents the portion of loss that is ceded to a reinsurer and $r(X)$ is the residual loss that is retained by the insurer. The functions f and r are respectively referred to as "indemnification function" (or "ceded loss function")