Insurance: Mathematics and Economics 67 (2016) 120–124

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

A note on some joint distribution functions involving the time of ruin

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ARTICLE INFO

Article history:
Received September 2015
Received in revised form December 2015
Accepted 16 December 2015
Available online 24 December 2015

Keywords:
Time of ruin
Deficit at ruin
Surplus prior to ruin
Joint distribution function
MAP risk model

ABSTRACT

In a recent paper, Willmot (2015) derived an expression for the joint distribution function of the time of ruin and the deficit at ruin in the classical risk model. We show how his approach can be applied to obtain a simpler expression, and by interpreting this expression by probabilistic reasoning we obtain solutions for more general risk models. We also discuss how some of Willmot's results relate to existing literature on the probability and severity of ruin.

1. Introduction

This note is concerned with finite time ruin problems and its starting point is a recent paper by Willmot (2015) who considers the classical risk model. Our results extend beyond the classical risk model, but we start here with a description of this model. As we make many references to Willmot (2015), we follow his notation. Thus, the insurer's surplus process is $\{U_t;\ t \geq 0\}$ where for $t \geq 0$, $U_t = u + ct - S_t$. Here $u \geq 0$ is the insurer's initial surplus, c is the rate of premium income per unit time (assumed to be received continuously), and $S_t = \sum_{i=1}^{N_t} Y_i$ denotes the aggregate claim amount up to time t, where $\{N_t; t \geq 0\}$ is a Poisson process with parameter λ and $\{Y_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables, with Y_i representing the amount of the ith claim. Let $P(y) = \Pr(Y_1 \leq y) = 1 - \bar{P}(y)$, $y \geq 0$, and let $p(y) = \frac{d}{dy}P(y)$. We assume that $c = (1+\theta)\lambda E(Y_1)$ where $\theta > 0$ is the premium loading factor.

The time of ruin is denoted T and is defined as $T = \inf\{t : U_t < 0\}$ with $T = \infty$ if $U_t \geq 0$ for all $t \geq 0$. Further let $|U_T|$ denote the deficit at the time of ruin and let U_{T-} denote the surplus immediately prior to ruin. We define the ultimate ruin probability as $\psi(u) = \Pr(T < \infty \mid U_0 = u) = 1 - \phi(u)$, the finite time ruin probability as $\psi(u,t) = \Pr(T \leq t \mid U_0 = u) = 1 - \phi(u,t)$, and what we call the t-deferred ruin probability as

$$\bar{\psi}(u,t) = \Pr(t < T < \infty | U_0 = u) = \psi(u) - \psi(u,t).$$

Similarly, we define probability and severity of ruin functions in finite and infinite time as $G(u,y) = \Pr(T < \infty,\ |U_T| \leq y | U_0 = u)$ and $G(u,y,t) = \Pr(T \leq t,\ |U_T| \leq y | U_0 = u)$, with the t-deferred probability and severity of ruin function being

$$\begin{aligned}\bar{G}(u,y,t) &= \Pr(t < T < \infty, |U_T| \leq y | U_0 = u) \\ &= G(u,y) - G(u,y,t).\end{aligned}$$

Let $F(y,t) = \Pr(S_t \leq y)$, with density function $f(y,t)$ for $y > 0$. Further, $F(0,t) = \Pr(N_t = 0) = e^{-\lambda t}$.

Willmot (2015) considers the equation

$$\frac{\partial}{\partial t}h(u,t) = c\frac{\partial}{\partial u}h(u,t) - \lambda h(u,t) + \lambda\int_0^u h(u-x,t)\,p(x)\,dx + \tau(u,t), \tag{1.1}$$

and the special case when $\tau(u,t) = \tau(u)$. He solves these equations using Laplace transform techniques, applying different approaches for the special case and the general case, and finds solutions for $\phi(u,t)$ and $G(u,y,t)$ using the special case solution. The solution for $\phi(u,t)$ is the well-known Prabhu's (1961) formula (as expected!), and the formula for $G(u,y,t)$ is new. However, this new formula is somewhat complicated, and it is not easy to give a probabilistic interpretation of it.

Equations similar to the special case of (1.1) (i.e. $\tau(u,t) = \tau(u)$) have a long history in the risk theory literature; see Arfwedson (1950), Prabhu (1961) and references therein. In this literature $\tau(u)$ has a specific form that relates to $h(u,t)$ having a particular ruin-theoretic interpretation. Problems previously studied include $h(u,t) = \bar{\psi}(u,t)$, and this points to an alternative approach to

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http://dx.doi.org/10.1016/j.insmatheco.2015.12.005
0167-6687/

finding a formula for $G(u, y, t)$, and to extensions, e.g. involving U_{T-}. The solution we obtain for $G(u, y, t)$ in the next section has a clear probabilistic interpretation which indicates how we can obtain solutions for more general risk models.

The main contributions in this note are a simple formula for $\bar{G}(u, y, t)$ and an extension in the next section, and in Section 3 a generalisation of this formula for $\bar{G}(u, y, t)$ to the MAP risk model. In Section 4 we consider an alternative formula for $G(u, y, t)$.

2. Joint distribution functions in the classical risk model

Willmot (2015) applies the result in his Theorem 2 to obtain expressions for $\phi(u, t)$ and $G(u, y, t)$. We now apply the same result to obtain $\psi(u, t)$. This will illustrate our earlier point about a complicated formula (since $\psi(u, t) = \lim_{y\to\infty} G(u, y, t)$), and will also serve to show that studying the t-deferred ruin probability provides a more straightforward approach to the problem of finding $\psi(u, t)$ and related probabilities from Eq. (1.1). Willmot's formula (17) yields the unsurprising result

$$\psi(0, t) = \int_0^t w(0, s)\, ds,$$

where $w(0, t) = \frac{d}{dt}\psi(0, t)$ is the (defective) density of T when $u = 0$ (see Dickson and Willmot, 2005 or Dickson, 2007), and for $u > 0$ we obtain

$$\begin{aligned}\psi(u, t) &= e^{-\lambda t}\,\alpha(u + ct) - \alpha(u)\\ &\quad + \int_0^{u+ct} \alpha(u + ct - x) f(x, t)\, dx\\ &\quad - c\int_0^t \psi(0, s) f(u + c(t - s), t - s)\, ds \qquad (2.1)\end{aligned}$$

where

$$\alpha(u) = \frac{1}{\lambda\theta E(Y_1)} \int_0^u \phi(u - x)\, \lambda\, \bar{P}(x)\, dx.$$

It is well known (e.g. Gerber, 1979) that

$$\phi(u) = \phi(0) + \frac{\lambda}{c}\int_0^u \phi(u - x)\, \bar{P}(x)\, dx,$$

giving

$$\alpha(u) = \frac{c}{\lambda\theta E(Y_1)}(\phi(u) - \phi(0)) = \frac{\phi(u)}{\phi(0)} - 1$$

since $\phi(0) = \theta/(1 + \theta)$. Thus, by Eq. (2.1) we obtain

$$\begin{aligned}\psi(u, t) &= e^{-\lambda t}\left(\frac{\phi(u + ct)}{\phi(0)} - 1\right) - \left(\frac{\phi(u)}{\phi(0)} - 1\right)\\ &\quad + \int_0^{u+ct}\left(\frac{\phi(u + ct - x)}{\phi(0)} - 1\right) f(x, t)\, dx\\ &\quad - c\int_0^t \psi(0, s) f(u + c(t - s), t - s)\, ds.\end{aligned}$$

This is a complicated formula, particularly when compared with writing $\psi(u, t)$ as the complement of Prabhu's formula for $\phi(u, t)$ (e.g. equation (5) in Willmot, 2015), nor does it appear to have a probabilistic interpretation. However, we can see from Prabhu's formula that

$$\begin{aligned}\psi(u, t) &= 1 - e^{-\lambda t} - \int_0^{u+ct} f(x, t)\, dx + c\int_0^t f(u + cs, s)\, ds\\ &\quad - c\int_0^t f(u + cs, s)\, \psi(0, t - s) ds.\end{aligned}$$

Equating these two expressions for $\psi(u, t)$ we obtain

$$\begin{aligned}\phi(u) &= e^{-\lambda t}\phi(u + ct) + \int_0^{u+ct} \phi(u + ct - x) f(x, t)\, dx\\ &\quad - c\int_0^t f(u + cs, s)\, \phi(0)\, ds, \qquad (2.2)\end{aligned}$$

which does have a clear probabilistic interpretation. Although this is not a useful formula for finding $\phi(u)$, it is useful in our subsequent development.

We now obtain a formula for $\bar{\psi}(u, t)$. The objective in obtaining this formula is not to provide a means of calculating $\bar{\psi}(u, t)$, since the most efficient approach to this is to deduct $\phi(u)$ from Prabhu's formula as

$$\bar{\psi}(u, t) = \psi(u) - \psi(u, t) = \phi(u, t) - \phi(u).$$

Rather, we seek a formula with a clear probabilistic interpretation that points the way to solutions to other problems. We see from Eq. (2.2) and Prabhu's formula that

$$\begin{aligned}\bar{\psi}(u, t) &= e^{-\lambda t}\,\psi(u + ct) + \int_0^{u+ct} \psi(u + ct - x) f(x, t)\, dx\\ &\quad - c\int_0^t \bar{\psi}(0, t - s) f(u + cs, s)\, ds \qquad (2.3)\end{aligned}$$

with $\bar{\psi}(0, t) = \psi(0) - \psi(0, t)$. Eq. (2.3) has a simple interpretation—the first two terms allow for ruin to occur from time t with the surplus at time t being $u + ct - x$, where $0 \le x < u + ct$ is the aggregate claim amount at time t, and the final term adjusts for realisations of the surplus process that have caused ruin to occur before time t, using the same arguments as in Prabhu's formula.

Building on this interpretation, it follows that

$$\begin{aligned}\bar{G}(u, y, t) &= e^{-\lambda t}\, G(u + ct, y)\\ &\quad + \int_0^{u+ct} G(u + ct - x, y) f(x, t)\, dx\\ &\quad - c\int_0^t \bar{G}(0, y, t - s) f(u + cs, s)\, ds. \qquad (2.4)\end{aligned}$$

This allows us to find $G(u, y, t)$ since the function $G(u, y)$ is well-documented (e.g. Gerber et al., 1987 and Drekic et al., 2004), and we can find $G(0, y, t)$ easily, for example from equation (24) of Willmot (2015) which is just

$$G(0, y, t) = \int_0^t \int_0^y w(0, x, s)\, dx\, ds,$$

where $w(0, x, s)$ is the (defective) joint density of the deficit at ruin (x) and the time of ruin (s) when $u = 0$, a formula for which is given in Dickson (2007).

We make two observations about formula (2.4). First, it seems a much simpler formula to apply to obtain $G(u, y, t)$ (as $G(u, y) - \bar{G}(u, y, t)$) than formula (22) of Willmot (2015). In particular, the first two terms of (2.4) are simpler than the first two terms of Willmot's (22), whilst the third terms in (2.4) and Willmot's (22) are very similar. If our objective is simply to obtain the joint distribution function of T and $|U_T|$ then (2.4) is a useful formula. However, it does not seem like a useful formula to obtain the joint density of T and $|U_T|$ (nor does formula (22) of Willmot, 2015); in particular it is not obvious to the author how to obtain the expression for the joint density of T and $|U_T|$ given in Dickson (2007) by differentiating (2.4).

Extending the previous interpretation, and now including the surplus prior to ruin, if we define

$$H(u, z, y) = \Pr(T < \infty,\ U_{T-} \le z,\ |U_T| \le y | U_0 = u),$$