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Risk capital allocation with autonomous subunits: The Lorenz set

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ABSTRACT

Risk capital allocation problems have been widely discussed in the academic literature. We consider a set of independent subunits collaborating in order to reduce risk: that is, when subunit portfolios are merged a diversification benefit arises and the risk of the group as a whole is smaller than the sum of the risks of the individual subunits. The question is how to allocate the risk capital of the group among the subunits in a fair way. In this paper we propose to use the Lorenz set as an allocation method. We show that the Lorenz set is operational and coherent. Moreover, we propose three fairness tests related directly to the problem of risk capital allocation and show that the Lorenz set satisfies all three tests in contrast to other well-known coherent methods. Finally, we discuss how to deal with non-uniqueness of the Lorenz set.

1. Introduction

When holding risky portfolios companies and financial institutions typically withhold a level of capital, which is invested safely and acts as a buffer against unfavorable events: so-called *risk capital*. Holding a certain amount of risk capital is not necessarily voluntary: in some cases regulation will directly require companies to withhold a minimum amount of risk capital in order to ensure against bankruptcy.

This paper considers fair allocation of risk capital in cases where risk is pooled over a group of independent subunits: either of the same company (for instance, if investment teams restrict their focus to different geographical areas, operate with different instruments, i.e., stocks vs. bonds, or simply have the right to choose the investment strategy autonomously); or, as part of an overall organizational/legal structure (for instance, joint ventures where firms have no prior historical collaboration, or where participants come to pool and spread risk, as known from e.g., Lloyd's[1]).

We ask how the risk capital of the overall organization should be allocated among the subunits given that these act as independent profit maximizing entities with their own corresponding risk profiles. This question is far from trivial since, unless perfectly correlated, the activities in different subunits will create diversification benefit when risks are pooled. Hence, a subunit may seem risky when looking only at the individual risk profile, but at the same time it can be useful in hedging other subunits' activities. Since the units are autonomous, though, we shall treat the situation as a pure cost sharing exercise: each subunit wants to minimize its share of the risk capital simply because they would rather invest in activities that give favorable returns instead of being required to withhold an amount with no or very low return. For illustrative purposes we assume that each subunit has its own financial portfolio. Nevertheless, in general the subunits could have other types of risky activities like e.g. insurance.

Typically cost allocation problems are modeled as a transferable utility game, see e.g., Hougaard (2009). The seminal paper on game theoretic risk capital allocation by Denault (2001) focuses on the well known Shapley and Aumann–Shapley cost allocation methods, and submits that a desirable allocation method should be, what is dubbed, *coherent*, i.e., that the resulting allocation should satisfy the stand-alone core conditions as well as a certain symmetry requirement. The primer makes sure that no coalition of subunits covers more than their own risk capital, while the latter ensures that equal subunits are treated equally. Since the Shapley value may fail the stand-alone core conditions it is not coherent while well-known solution concepts like the *nucleolus* (Schmeidler, 1969) and the *Aumann–Shapley value*[2] (Aumann and Shapley, 1974) both are examples of coherent allocation rules.

Several other papers analyze risk capital allocation from a game theoretic and axiomatic viewpoint. For instance, recently Chen et al. (2013) consider the systemic risk of an entire economy

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1 www.lloyds.com.

2 Also known as the Euler or gradient method in the finance literature.

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and how to attribute risk to individual companies. Most other papers consider risk capital allocation between subunits at a company level. For example, Tsanakas and Barnett (2003) and Boonen et al. (2013) focus on the Aumann–Shapley rule and in the latter case, generalized weighted versions of this rule. Gulick et al. (2012) suggest to use a version of the nucleolus with a different notion of coalitional excess. Balog et al. (0000) and Homburg and Scherpereel (2008) focus on comparisons of several relevant allocation rules, including the Shapley value, the nucleolus and the Cost-Gap and β-method. In Homburg and Scherpereel (2008) they specifically look at the risk measure value-at-risk (VaR, which is not coherent, see Artzner et al., 1999) and demonstrate by an experiment that decision makers tend to disregard stand-alone core conditions and prefer simple methods like the β-method. Csóka et al. (2009) consider the formal relation between the class of risk allocation games and the class of totally balanced as well as exact games. In Csóka and Herings (2014) liquidity considerations are included.

In this paper we show that the current methods used for risk capital allocation can yield quite unpalatable results, especially when the primal goal of risk pooling deviates from pure profit maximization under a certain risk appetite.

Since we consider the units as autonomous the stand-alone core conditions are particularly compelling. The stand-alone core is considered a fundamental fairness requirement of any allocation method—especially when the problem itself is balanced (i.e., the core is non-empty) as in the case of risk capital allocation using a coherent risk measure (e.g., expected shortfall). We agree that relevant allocation methods should indeed be *coherent* in the sense of Denault (2001).[3] Yet, among the coherent methods we focus on egalitarian allocations. In particular, we suggest to apply the Lorenz set (i.e., the set of Lorenz undominated allocations in the core) for risk capital allocation. Basically this solution concept looks for the most equally distributed allocations of risk capital subject to the stand-alone core conditions, and it is well-known in game theory (see e.g., Dutta and Ray, 1989 and Hougaard et al., 2001) but apparently it has not been analyzed in connection with risk capital allocation. We demonstrate that the Lorenz set is coherent (with a straightforward generalization of Denault's definition of coherence to cover set valued solutions) and has further advantages over alternative well-known coherent solutions which are related directly to the problem of risk capital allocation.

We show that, contrary to the well-known methods, the Lorenz set ensures that; (1) every subunit with a risky portfolio holds a strictly positive level of risk capital if the stand-alone core conditions allow such allocation; (2) changes in a subunit's portfolio that benefit the overall risk situation will never be punished by allocating too much extra risk capital to that subunit; (3) when the risk is eliminated no subunit subsidizes another subunit. These three conditions are dubbed Fairness tests I, II and III.

Although the Lorenz set is not a singleton it is still operational as a solution concept. An algorithm that finds one Lorenz undominated allocation followed by search on adjacent core facets will enable a full computation of the Lorenz set in finite time, see Smilgins (2016).

The paper is organized as follows: In Section 2 we set up the model. In Section 3 we define the Lorenz set and record a few useful properties. In Section 4 we submit three fairness tests directly related to risk capital allocation and demonstrate that these properties are all satisfied by the Lorenz set but not by any of conventional methods from the literature on risk capital allocation (i.e., the Shapley value, the Cost-Gap method, the nucleolus and the Aumann–Shapley value). Section 5 closes with a few final remarks.

[3] Even if the subunits are forced to stay part of the company (at least in the short run) and hence cannot threat to block the cooperation the stand-alone core conditions are still relevant since they ensure that no coalition of subunits is subsidized by other subunits.

2. The model

In this paper we consider the allocation of risk capital along the lines of Denault (2001). Imagine a set of n independent subunits, denoted by $N = \{1, \ldots, n\}$, wanting to pool their risks as a group. Each subunit $i \in N$ has its own portfolio, and we assume that the other subunits' portfolios are unknown to i.

At present time, say t_0, we know exactly how much each subunit's portfolio is worth. However, at a specific point of time in the future, say t_1, the net worth of the n portfolios are unknown. Denote by V the set of admissible portfolios and let subunit i's payoff be modeled by a random variable $X_i = r_i A_i \in V$ representing the net worth of i's investment of A_i dollars in a portfolio with a stochastic return r_i between time periods t_0 and t_1. Let $X = \{X_1, \ldots, X_n\}$ denote the companies payoff profile and let $X(S) = \sum_{i \in S} X_i$ be the payoff of the pooled portfolio of coalition $S \subseteq N$. That is, $X(N)$ is the payoff of the group as a whole at time t_1.

Risk is quantified by a *risk measure* $\rho : V \to \mathbf{R}$. In the following analysis we will always assume that the risk measure involved is a so-called *coherent* risk measure in the sense of Artzner et al. (1999). In particular, all our examples will use *Expected Shortfall (ES)* with a degree of confidence of 5% as risk measure and ignore all kinds of transaction costs for simplicity, see e.g., Artzner et al. (1999) or Acerbi and Tasche (2002). Expected Shortfall is coherent both in case of continuous and discrete distributions of returns. The interpretation is most straightforward in the former case, where ES indicates, for each portfolio, the amount of riskless capital that should be withheld by the company in order to be able to cover expected losses given that the payoff is below a certain threshold value.

To withhold riskless capital can be considered as a cost. Thus, for each coalition of subunits $S \subseteq N$, the cost associated with the payoff $X(S)$ of the pooled portfolio is defined as

$$c(S) = \rho(X(S)), \tag{1}$$

with $c(\varnothing) = 0$ per definition. We say that $c(S)$ is the *risk capital* of coalition $S \subseteq N$. In particular, $c(N)$ is total risk capital of the group that has to be allocated among the n subunits. As such, the problem can be modeled as a transferable utility (TU) game, see e.g., Peleg and Sudhölter (2003).

Denote by (N, c), where N is the set of subunits and c is the cost function determined by (1), a *risk capital allocation problem*. Let Γ be the set of all such problems.

Let $Y(N, c) = \{y \in \mathbf{R}^N \mid \sum_{i \in N} y_i = c(N)\}$ be the set of possible allocations of the total risk capital $c(N)$. A *solution* on Γ is a mapping σ which associates with each problem $(N, c) \in \Gamma$ a subset $\sigma(N, c)$ of $Y(N, c)$.

One such well-known solution is *the core* given by

$$\mathcal{C}(N, c) = \left\{ y \in Y(N, c) \mid \sum_{i \in S} y_i \le c(S) \text{ for all } S \subseteq N \right\}. \tag{2}$$

The core consists of allocations of risk capital for which no coalition of subunits pay more than the risk capital associated with their pooled portfolios and thereby S does not subsidize other subunits. Mathematically speaking the core is an $n-1$ dimensional polytope, i.e., it is a closed and convex set with facets that are polytopes themselves.

By Theorem 4 in Denault (2001) it is known that if the risk measure ρ is coherent then the core of the associated risk capital problem is non-empty (i.e., the problem (N, c) is *balanced* by the