Insurance: Mathematics and Economics 60 (2015) 19–28

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Analytical pricing of vulnerable options under a generalized jump–diffusion model

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ARTICLE INFO

Article history:
Received March 2014
Received in revised form October 2014
Accepted 11 October 2014
Available online 30 October 2014

JEL classification:
G13
G22
D52

Keywords:
Vulnerable options
Reduced form
Esscher–Girsanov transform
Generalized jump
Credit risk

ABSTRACT

In this paper we propose a model to price European vulnerable options. We formulate their credit risk in a reduced form model and the dynamics of the spot price in a completely random generalized jump–diffusion model, which nests a number of important models in finance. We obtain a closed-form price for the vulnerable option by (1) determining an equivalent martingale measure, using the Esscher transform and (2) manipulating the pay-off structure of the option four further times, by using the Esscher–Girsanov transform.

1. Introduction

Vulnerable options are contingent claims on defaultable instruments, subject to their issuer's default risk. Financial institutions actively trade derivative contracts with their corporate clients, as well as with other financial institutions in over-the-counter (OTC) markets. The absence of a clearing house in the OTC market imposes the counter-party credit risk on the holder of these contracts.

There are two classes of models that capture credit default risks. One class is structural (or Merton) credit models, which are microeconomic models of the firm's capital structure. The structural approach was first introduced in Merton (1974), where a single-period model was utilized to derive the default probability from the random variation in the unobservable value of the firm's assets. These models were further developed by Black and Cox (1976), as well as Longstaff and Schwartz (1995). The other major class of credit risk modeling research focuses on reduced-form models of default, which assumes a firm's default time is inaccessible or unpredictable and driven by a default intensity that is a function of latent state variables. Jarrow and Turnbull (1995), Duffie and Singleton (1999) and Hull and White (2000) present detailed explanations of several well known reduced-form modeling approaches. Due to their mathematical tractability, these models have become very popular amongst practitioners.

The first paper that considered the pricing of vulnerable options was Johnson and Stulz (1987), where they assumed the option as the only liability of the issuer. They extended Merton's corporate bond default model to price vulnerable options; therefore, in their model the default occurs in the event that the total payout obligation of the option at the expiry exceeds the total value of the issuer's assets. Their model was extended by Klein (1996), allowing the option writer to have other liabilities which rank equally with payments under the option. Additionally, Klein (1996) assumed a constant default barrier as a constant approximation to the optimal default decision. This model was further extended in Klein and Inglis (2001) by incorporating the potential liability of the written option into the default boundary.

One modeling challenge for pricing vulnerable options is the additional credit risk process, which complicates the mathematical tractability of the models. Many researchers have explored different technical facets of pricing vulnerable options. For instance, Klein and Inglis (1999) derived an analytical price for the vulnerable option using the partial differential equation (PDE) approach, Hui et al. (2003) considered the stochastic default barrier, and Tian et al. (2013) extended the work of Klein and Inglis (2001) by incorporating the jump risk in the underlying asset. The latter paper provided a pricing model for vulnerable options, which face not

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http://dx.doi.org/10.1016/j.insmatheco.2014.10.007
0167-6687/

only default risk but also rare shocks encountered by the underlying asset and the assets of the counter-party.

In this paper, we investigate the valuation of European vulnerable options under a completely random generalized jump–diffusion model for the underlying asset, which nests a number of important and popular models in finance. For example, Merton jump–diffusion model (Merton, 1973), the generalized gamma (GG) process (including the scale distorted and power distorted versions), the variance gamma (VG) process (Madan et al., 1998) and the CGMY process (Carr et al., 2002). We adopt a reduced form formulation for the default risk, chosen in consistence with the dynamics of the stochastic short-rate. We, additionally, consider the underlying asset, the default risk, and the short-rate process to be correlated, in order to develop a more realistic modeling framework. Therefore, our model simultaneously captures both rare events of the stock prices, as well as the default risk of the issuer in the finance market.

The highlight of this paper is the analytical price for the vulnerable option. We demonstrate the usefulness of the method of change of measures to achieve an analytical solution for our underlying problem. We first calculate the market price of risk via the determination of an equivalent martingale measure, using the Esscher transform. Then we manipulate the pay-off structure of the contingent claim four further times, by using the Esscher–Girsanov transform. A thorough discussion of these probability measure transforms and applications in derivatives pricing is presented in Goovaerts and Laeven (2008). We shall emphasize that in the present paper the change of measure by the Esscher–Girsanov transform is adopted as a computational device to achieve a closed-form solution for the contingent claim, in contrast to Goovaerts and Laeven (2008) or Badescu et al. (2009), who used it as a pricing device.

The paper is organized as follows: Section 2 focuses on the model descriptions of the dynamics for the underlying asset, the stochastic short-rate, and the credit risk. In this section we also explain the derivations for the equivalent martingale measure, used in the pricing methodology. Section 3 discusses the analytical pricing of European vulnerable options. Section 4 provides a thorough discussion of some parameter specifications for the generalized jump–diffusion model. Section 5 looks at the numerical analysis of the model, and Section 6 concludes the paper.

2. Model description

Consider a continuous time model with two primary traded assets, namely, a zero coupon bond and a stock. First, we shall present the stock price S assumptions and derive the risk neutral dynamics via the Esscher transform. Then we describe the model assumptions for the short-rate and the credit risk.

Fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the historical probability measure. Let $\mathcal{T}$ denote the time index set $[0, \infty)$ and $\mathcal{B}(\mathcal{T})$ the Borel field on $[0, \infty)$. Write $\mathcal{R}^+ = (0, \infty)$ and $\mathcal{B}(\mathcal{R}^+)$ the Borel field on $\mathcal{R}^+$.

For the dynamics of the underlying asset, we consider a kernel-biased completely random jump–diffusion model. James (2002, 2005) proposed a kernel-biased representation of completely random measures, which provides flexible modeling for different types of finite and infinite jump activities. The approach is an amplification of Bayesian techniques developed by Lo and Weng (1989) for gamma-Dirichlet processes.

For each $U \in \mathcal{B}_0$, let $N(., U)$ denote a Poisson random measure. Write $N(dt, dz)$ for the differential form of measure $N(t, U)$. Let $\varrho(dz|t)$ denote a Lévy measure on the space depending on t; η is a σ-finite measure on $\mathcal{T}$. As in James (2005), the existence of the kernel-biased completely random measure is ensured by supposing an arbitrary positive function on $\mathcal{R}^+$, $h(z).\varrho_i$ and η are selected in such a way that for each bounded set $\mathcal{B}$ in $\mathcal{T}$,

$$\int_{\mathcal{B}} \int_{\mathcal{R}^+} \min(h(z), 1)\varrho(dz|t)\eta(dt) < \infty.$$

Assume that the intensity measure $\nu_{X_t}(dt, dz)$ for the Poisson random measure $N(dt, dz)$ is given by

$$\nu(dt, dz) := \varrho(dz|t)\eta(dt),$$

where the intensity function $\varrho(dz|t)$ of the jump size can depend on time t. In this case, jump sizes and jump times are correlated.

Let $\widetilde{N}(dt, dz)$ denote the compensated Poisson random measure defined by

$$\widetilde{N}(dt, dz) = N(dt, dz) - \varrho(dz|t)\eta(dt).$$

In addition, define a kernel-biased completely random measure on $\mathcal{T}$ as follows:

$$\kappa(dt) := \int_{\mathcal{R}^+} h(z)N(dt, dz),$$

which is a kernel-biased Poisson random measure $N(dt, dz)$ over the state space of the jump size $\mathcal{R}^+$ with the mixing kernel function $h(z)$. In general, we can replace the Poisson random measure with a random measure and choose some potentially exotic functions for $h(z)$ to generate different types of finite and infinite jump activities. So define a compensator of $\kappa(dt)$, as follows:

$$\int_{\mathcal{R}^+} h(z)\nu(dt, dz) = \int_{\mathcal{R}^+} h(z)\varrho(dz|t)\eta(dt).$$

Then let $\widetilde{\kappa}(dt)$ denote the compensated completely random measure on $\mathcal{T}$ corresponding to $\kappa(dt)$, such that:

$$\widetilde{\kappa}(dt) := \int_{\mathcal{R}^+} h(z)\widetilde{N}(dt, dz).$$

Hence,

$$\widetilde{\kappa}(dt) = \kappa(dt) - \int_{\mathcal{R}^+} h(z)\nu(dt, dz).$$

Then, suppose that the stock price S_t satisfies the following Stochastic Differential Equation (SDE) under $\mathbb{P}$:

$$\frac{dS_t}{S_{t-}} = \mu_t dt - \int_{\mathcal{R}^+} h(z)\varrho(dz|t)\eta(dt) + \sigma_{1t} dW_{1t} + \int_{\mathcal{R}^+} (e^{h(z)} - 1)\widetilde{N}(dt, dz), \quad (1)$$

where W_{1t} is standard Brownian motion, $\{\mu_t\}_{t\in\mathcal{T}}$ is the stock appreciation rate and $\{\sigma_{1t}\}_{t\in\mathcal{T}}$ is the stochastic volatility of the stock. Let $Y_t := \ln(S_t/S_0)$ be the return process, then by Ito's lemma

$$dY_t = \left(\mu_t - \frac{1}{2}\sigma_t^2\right)dt + \sigma_t dW_{1t} + \int_{\mathcal{R}^+} h(z)\widetilde{N}(dt, dz). \quad (2)$$

Let $\mathcal{F}_t^S = \sigma(S_s, 0, \le s \le t)$ which is the σ-field generated by the price process S_t. In Section 2.2, we will also define the filtration for the interest rate process $(\mathcal{F}_t^r)$, the filtration for the default process $(\mathcal{F}_t^\lambda)$, and the filtration for the event of default $(\mathcal{H}_t)$.

2.1. Equivalent martingale measure via Esscher transform

The market described in this paper is incomplete. Therefore, there is more than one equivalent martingale measure. Here, we describe how to determine an equivalent martingale measure by the Esscher transform.