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On the efficient utilisation of duration

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ABSTRACT

In this article we present a new approach to estimate the change of the present value of a given cashflow pattern caused by an interest rate shift. Our approximation is based on analysing the evolution of the present value function through a linear differential equation. The outcome is far more accurate than the standard approach achieved by a Taylor expansion. Furthermore, we derive an approximation formula of second order that produces nearly accurate results. In particular, we prove that our method is superior to any known alternative approximation formula based on duration. In order to demonstrate the power of this improved approximation we apply it to coupon bonds, level annuities, and level perpetuities. We finally generalise the approach to a non-flat term structure. As for applications in insurance, we estimate the change of the discounted value of future liabilities due to a proportional shift in the set of capital accumulation factors. These findings are of particular importance to capital adequacy calculations with respect to interest rate stress scenarios that are part of regulatory solvency requirements.

1. Introduction

The time value of money depends on the characteristics of the particular cash flow: its term, the pattern of payments, and the discount rates applied to the payments. In this article, we evaluate the evolution of the present value of a given cash flow along with interest rates changes. Essentially, duration is the link between the change in the present value of a given cash flow on the one hand and the change in interest rates on the other hand. In fact, duration is widely used in the finance industry as an indicator of the price change resulting from a change in yield to maturity (Bierwag and Fooladi, 2006).

Macauley (1938) introduced a measure called duration as the expected lifetime for bonds. He suggested calculating the weighted average of the time to each scheduled payment. Duration may thus be interpreted as the expected term or weighted average term. The weight associated with each payment is the present value of that payment divided by the bond price. Following Hicks (1939) duration is frequently referred to as a measure for price volatility. He analysed the price elasticity of a cash flow stream. As a consequence, the change in the value of an asset due to an interest rate change is closely related to the duration of the asset. In this respect, it is noteworthy that in the 19th century British actuaries studied the relationship between the price of an annuity and its derivative with regard to the yield to maturity (see Lidstone, 1895, for instance). Samuelson (1945) rediscovered duration by considering the derivative of the bond price with respect to yield to maturity. Redington (1952) used duration for achieving immunisation against interest rate changes. These four scientists are widely regarded as the pioneers of the concept of duration and its applications in finance (Bierwag and Fooladi, 2006).

The approximation of the present value of a given cash flow based on the concept of duration has a number of practical merits in finance. To begin with, it provides an insight concerning price sensitivity of fixed income securities. In particular, bond portfolios have a mixed cash flow pattern which renders it complex to accurately calculate the price change for a change in interest rates. Likewise, duration allows for analysing the impact of interest rate changes on a company's assets and liabilities. Even with the brute force of today's computers the calculations needed for exact values may turn out to be arduous. Last but not least, the duration approach is helpful for assessing the impact of interest rate shocks for solvency purposes. An improved approximation formula is

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needed in this context in order to provide for greater accuracy than the standard methodology found in the literature.

The key idea of our approach is to consider a linear differential equation for the evolution of the present value function. We tackle its solution by making an appropriate simplification to start with. With a view to this, our article is structured as follows. We introduce basic notations in Section 1. In Section 2 we derive two approximation formulae of first and second order, respectively. In Section 3 we compare our method to others from a theoretical point of view. In particular, we prove that our formula is superior to other approximations. In Section 4 we conduct a number of numerical analyses for bonds, level annuities and level perpetuities. Finally, in Section 5, we generalise our approximation formula to encompass changes in the term structure of interest rates. It turns out that the effect of a proportional shift of interest rates on the present value can be easily approximated. We illustrate this finding with respect to the approximation of claim reserves in life, motor, and personal accident insurance.

1.1. Notations

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ be a cash flow representing n payments that are due in arrears and uniformly distributed in time. Further, let i be the flat interest rate per period that is applicable to discount all future payments. Then, the present value is defined as the time value of money:

$$P(i) = \sum_{k=1}^{n} \alpha_k (1+i)^{-k}. \tag{1}$$

Moreover, the duration of this cash flow is given by

$$D(i) = -(1+i)\frac{P'(i)}{P(i)}. \tag{2}$$

Usually, $P(i_0)$ and $D(i_0)$ are known for a given initial interest rate i_0. Next, we are interested in the value $P(i)$ in a neighbourhood of i_0. The classic approach employs Taylor's expansion. Thus, the present value is approximately

$$P(i) \approx P(i_0) + P'(i_0)\cdot(i-i_0) = P(i_0) - \frac{D(i_0)\cdot P(i_0)}{1+i_0}(i-i_0). \tag{3}$$

Note that this linear approximation is valid only for small changes in interest rates. For a big change in the yield to maturity the approximation error becomes very large.

The improved classic approximation takes into account the second derivative of the present value with respect to the interest rate:

$$P(i) \approx P(i_0) + P'(i_0)\cdot(i-i_0) + \frac{1}{2}P''(i_0)\cdot(i-i_0)^2. \tag{4}$$

Thereby, convexity $C(i_0)$, defined by

$$C(i_0) = \frac{P''(i_0)}{P(i_0)} \tag{5}$$

is supposed to be known, too, for an initial yield i_0. Finally, dispersion $M^2(i_0)$, defined by

$$M^2(i_0) = (1+i_0)^2 C(i_0) - D^2(i_0) - D(i_0) \tag{6}$$

is related to the derivative of duration:

$$D'(i_0) = \frac{-M^2(i_0)}{1+i_0} = \frac{D^2(i_0)+D(i_0)}{1+i_0} - (1+i_0)C(i_0). \tag{7}$$

2. The improved approximation

We focus on the derivative of the present value function with respect to interest rate. If duration $D(i)$ was known for all interest rates i the present value function $P(i)$ could be determined by solving a linear differential equation. Since duration is usually known for a given yield i_0 only we make a suitable approximation in the first place and, subsequently, derive the approximated present value function $P^{\text{app}}(i)$.

2.1. The approximation formula of first order

According to formula (2) we know that the evolution of the presented value is governed by a linear differential equation:

$$P'(i) = -\frac{D(i)}{1+i}P(i). \tag{8}$$

Notably, both present value $P(i_0)$ as well as duration $D(i_0)$ are only known for a given interest rate i_0. By replacing $D(i)$ by $D(i_0)$ in formula (8) we find an approximation for the evolution of the present value:

$$P'(i) \approx -\frac{D(i_0)}{1+i}P(i). \tag{9}$$

The differential equation presented in (9) is solved by

$$P(i) = c(1+i)^{-D(i_0)}$$

with $c \in \mathbb{R}$ being a constant since it is true that

$$\begin{aligned} P'(i) &= c\,(-D(i_0))\,(1+i)^{-D(i_0)-1} \\ &= -\frac{D(i_0)}{1+i}c(1+i)^{-D(i_0)} = -\frac{D(i_0)}{1+i}P(i). \end{aligned}$$

Applying the initial condition $P(i_0)$ at $i = i_0$ we are able to determine the constant c:

$$P(i_0) = c(1+i_0)^{-D(i_0)} \Leftrightarrow c = \frac{P(i_0)}{(1+i_0)^{-D(i_0)}}.$$

It follows that

$$\begin{aligned} P(i) &= c(1+i)^{-D(i_0)} = \frac{P(i_0)}{(1+i_0)^{-D(i_0)}}(1+i)^{-D(i_0)} \\ &= P(i_0)\left(\frac{1+i_0}{1+i}\right)^{D(i_0)}. \end{aligned}$$

In order to distinguish different approximations we denote this approximation by

$$P^{\text{app}}(i) = P(i_0)\left(\frac{1+i_0}{1+i}\right)^{D(i_0)}. \tag{10}$$

In order to evaluate goodness of fit we compare formula (8) and formula (9). We first note that duration decreases for increasing interest rates, i.e. $D(i) < D(i_0)$ for $i > i_0$. Consequently, the derivative of the exact present value function is greater than the derivative of the approximated function: $P'(i) > P^{\text{app}'}(i)$ for $i > i_0$. Note that the derivative of the present value function is always negative. Since both functions take the same value at i_0, namely $P(i_0) = P^{\text{app}}(i_0)$, it follows because of continuity that $P(i) > P^{\text{app}}(i)$ for $i > i_0$. An analogous argument holds for $i < i_0$. Therefore, we proved in general that the approximated value is always bounded above by the exact present value:

$$P^{\text{app}}(i) \le P(i). \tag{11}$$

Notably, quantitative analysts who are risk averse find it conservative and desirable that the approximated present value never exceeds the exact value.