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Optimal reinsurance under risk and uncertainty

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ABSTRACT

This paper deals with the optimal reinsurance problem if both insurer and reinsurer are facing risk and uncertainty, though the classical uncertainty free case is also included. The insurer and reinsurer degrees of uncertainty do not have to be identical. The decision variable is not the retained (or ceded) risk, but its sensitivity with respect to the total claims. Thus, if one imposes strictly positive lower bounds for this variable, the reinsurer moral hazard is totally eliminated.

Three main contributions seem to be reached. Firstly, necessary and sufficient optimality conditions are given in a very general setting. Secondly, the optimal contract is often a bang–bang solution, *i.e.*, the sensitivity between the retained risk and the total claims saturates the imposed constraints. Thirdly, the optimal reinsurance problem is equivalent to other linear programming problem, despite the fact that risk, uncertainty, and many premium principles are not linear. This may be important because linear problems may be easily solved in practice, since there are very efficient algorithms.

1. Introduction

Since Borch (1960) and Arrow (1963) published their celebrated seminal papers, the optimal reinsurance problem has been addressed by many authors and under many different risk measurement methods and premium principles. Recent approaches are, amongst many others, Kaluszka (2005), Cai and Tan (2007), Chi and Tan (2013), Tan and Weng (2014) or Cheung et al. (2014). Usually, researchers consider the insurer point of view, though the reinsurer viewpoint may be also incorporated (Cai et al., 2012, Cui et al., 2013, etc.). An interesting survey about the State of the Art in 2009 may be found in Centeno and Simoes (2009).

All the papers above assume that the statistical distribution of claims is known. Nevertheless, measurement errors or lack of complete information may provoke discrepancies between the real and the estimated probabilities of the states of nature, generating uncertain (also called ambiguous) frameworks. Actuarial and financial literature is recently paying significant attention to those cases where the probabilities of the scenarios are not totally known. Interesting examples are, among many others, portfolio management (Zhu and Fukushima, 2009), equilibrium in asset markets (Bossaerts et al., 2010), optimal stopping (Riedel, 2009) and insurance pricing (Pichler, 2014).

The first objective of this paper is to incorporate ambiguity in the optimal reinsurance problem, though many results will be also new in the uncertainty free setting. Both insurer and reinsurer may be ambiguous, but their degrees of ambiguity do not have to be identical. Since the reinsurer information about the reinsured set of policies could be lower than the information of the insurer, it seems natural to assume that the reinsurer ambiguity is higher, but we will not impose this hypothesis because we will not need it. According to the empirical evidence and the famous Ellsberg paradox, agents usually reflect ambiguity aversion, so we will accept this assumption in our analysis. Though there are other recent approaches (Maccheroni et al., 2006), the worst-case principle properly incorporates the ambiguity aversion (Gilboa and Schmeidler, 1989), and therefore our analysis will deal with this principle when considering the insurer expected wealth, the insurer global risk

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(integrating uncertainty too) and the reinsurer premium principle. Actually, all of the papers above deal with ambiguity by means of a worst-case approach.

Stop-loss or closely related contracts frequently solve the optimal reinsurance problem. These solutions have been often criticized by both theoretical researchers and practitioners. In practice, reinsurers will rarely accept these solutions due to the lack of incentives of the insurer to verify claims beyond some thresholds. Our second objective will be to overcome this caveat. Consequently, the insurer decision variable will be the (almost everywhere) mathematical derivative of the retained risk with respect to the global claims, rather than the retained risk itself. With this modification we can impose positive lower bounds to this decision variable, and therefore contracts reflecting spreads with null derivative (flat behavior of the retained risk with respect to the global claims) become unfeasible. In other words, the usual reinsurer moral hazard is eliminated with this approach.

The paper is organized as follows. Section 2 will present the general framework, the set of priors, the properties of the insurer risk measure (integrating uncertainty), the properties of the reinsurance premium principle (which may incorporate the reinsurer uncertainty) and the general optimal reinsurance problem we are going to deal with. We will point out how our approach contains most of the usual cases and extends them all if ambiguity arises. Section 3 will be devoted to dealing with two dual approaches. Theorems 4 and 5 will provide us with two alternative dual problems, as well as two different families of necessary and sufficient optimality conditions. It is worth to point out that one of the duals is linear.

The optimality conditions will generate two different ways permitting us to linearize the optimal reinsurance problem. The first one is the introduction of a linear optimization problem generated by a dual solution. This method will allow us to prove Theorem 7 in Section 4, which will give sufficient conditions to guarantee that the optimal retention is a bang–bang contract, *i.e.*, a contract such that the derivative of the retained risk with respect to the total one saturates the imposed constraints. A clear consequence is that in many classical approaches one must find stop-loss or closely related optimal contracts. In our less restrictive framework the optimal retention will be often a bang–bang solution.

Section 5 explores a second linearization procedure. In order to simplify the exposition the focus is on the Robust Conditional Value at Risk (robust *CVaR*) as an insurer risk/ambiguity measure and a reinsurer instrument to generate the premium principle. The method applies for much more situations, but selecting one important case we significantly shorten the paper. Furthermore, the *CVaR* is becoming more and more popular among researchers and practitioners due to its interesting properties (Ogryczak and Ruszczynski, 2002).

Since one of the two duals of Section 3 is linear, we will construct the double-dual (dual of the dual) optimal reinsurance problem in Section 5, which is linear too. We will prove that the solution of the double-dual will directly lead to the optimal reinsurance contract. This seems to be a very important property because there are many efficient algorithms solving linear problems in both finite-dimensional and infinite-dimensional frameworks (Anderson and Nash, 1987). Besides, linear problems often lead to extreme solutions, which explain why the nonlinear optimal reinsurance problem may be solved by a bang–bang retention.

The last section of the paper summarizes the most important conclusions, emphasizing the two main novelties (uncertainty introduction and moral hazard elimination) and the three main contributions (necessary and sufficient optimality conditions, bang–bang solutions and double-dual linear problems).

Throughout the paper we will need several mathematical notions about topological spaces, Banach and Hilbert spaces, weak convergences, etc. Some of them will be briefly summarized, but further discussions may be found in Luenberger (1969), Kelly (1975), Rudin (1973, 1987), or Anderson and Nash (1987), amongst others.

2. The optimal reinsurance problem

Consider the random total cost (claims) that an insurer will pay within a period $[0, T]$. This cost cannot achieve negative values, and the existence of an upper bound M is obvious too (claims cannot be higher than the value of the insured goods). Thus, we can deal with the Borel σ-algebra $\mathcal{B}$ of $[0, M]$ in order to represent the information that will be available at T about claims. Furthermore, if we assume that the insurer is ambiguous (or reflects uncertainty) with respect to the probabilities associated with the cost of claims, then her/his uncertainty level may be given by a set $\mathcal{P}_U^0$ of probability measures (or set of priors) on $\mathcal{B}$.

2.1. The set of priors

Next, let us give the main properties that $\mathcal{P}_U^0$ will have to satisfy. Denote by $\mathcal{P}$ the set of probability measures on $\mathcal{B}$ and fix $\mathbb{P}_0 \in \mathcal{P}$. Consider the Hilbert space $L^2(\mathbb{P}_0)$, which is composed of those random variables x whose square has finite expectation with respect to $\mathbb{P}_0$ and which is endowed with the norm

$$\|x\|_{(2,\mathbb{P}_0)} = \left[\int_0^M x^2(t)\, d\mathbb{P}_0(t)\right]^{1/2}, \quad x \in L^2(\mathbb{P}_0). \tag{1}$$

Similarly, consider the usual Lebesgue measure on $[0, M]$ and the classical Hilbert space $L^2[0, M]$.[1] The well-known norm of $L^2[0, M]$ is given by

$$\|x\|_2 = \left[\int_0^M x^2(t)\, d(t)\right]^{1/2}, \quad x \in L^2[0, M]. \tag{2}$$

We will assume the existence of $R \in L^2(\mathbb{P}_0)$, $R \geq 1$, such that

$$\mathcal{P}_U^0 = \left\{p \in \mathcal{P};\ 0 \leq \frac{dp}{d\mathbb{P}_0} \leq R\right\}, \tag{3}$$

$\frac{dp}{d\mathbb{P}_0}$ denoting the Radon–Nikodym derivative of p with respect to $\mathbb{P}_0$. If the operator $\mathbb{E}_p$ denotes mathematical expectation with respect to a probability measure p, and, in particular, $\mathbb{E}_{\mathbb{P}_0}$ denotes mathematical expectation with respect to $\mathbb{P}_0$, the set of priors (3) may be also given by

$$\mathcal{P}_U = \left\{f \in L^2(\mathbb{P}_0);\ 0 \leq f \leq R,\ \mathbb{E}_{\mathbb{P}_0}(f) = 1\right\}, \tag{4}$$

since we can obviously identify every $\mathbb{P}_0$-continuous probability measure $p \in \mathcal{P}_U^0$ with its Radon–Nikodym derivative $f = \frac{dp}{d\mathbb{P}_0}$. In other words, the insurer uncertainty will be identified with a subset of the interval $[0, R] \subset L^2(\mathbb{P}_0)$. Though it is an abuse of language, we will also say the (4) is the set of the insurer priors.

The set of priors (3) is very general. Firstly, if $R = 1$ then the non-ambiguous (or uncertainty free) case will be included in our

[1] The role of $L^2(\mathbb{P}_0)$ and of $L^2[0, M]$ may be plaid by other spaces of random variables such as $L^q(\mathbb{P}_0)$ and $L^q[0, M]$ with $q > 1$. Most of the results of this paper may be extended. Nevertheless, to the best of our knowledge, Theorem 10 in Section 5 cannot be extended. The presented proof does not apply for $q \neq 2$. The reason is that $L^q(\mathbb{P}_0)$ and $L^q[0, M]$ are not Hilbert spaces if $q \neq 2$, which implies that conditional expectations cannot be interpreted as orthogonal projections. Since the constraint $q = 2$ simplifies the mathematical exposition and does not seem to restrict the real life applications of this paper, we have decided to impose it, though it would have been sufficient to introduce this constraint in Theorem 10.