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Optimal retention for a stop-loss reinsurance with incomplete information

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ABSTRACT

This paper considers the determination of optimal retention in a stop-loss reinsurance. Assume that we only have incomplete information on a risk X for an insurer, we use an upper bound for the value at risk (VaR) of the total loss of an insurer after stop-loss reinsurance arrangement as a risk measure. The adopted method is a distribution-free approximation which allows to construct the extremal random variables with respect to the stochastic dominance order and the stop-loss order. We derive the optimal retention such that the risk measure used in this paper attains the minimum. We establish the sufficient and necessary conditions for the existence of the nontrivial optimal stop-loss reinsurance. For illustration purpose, some numerical examples are included and compared with the results yielded in Theorem 2.1 of Cai and Tan (2007).

1. Introduction

The importance of managerial decisions related to optimal reinsurance has received considerable attention in actuarial literature. It usually involves formulating an optimization problem and obtaining its optimal solution under certain criterion. Recently, optimization criteria based on tail risk measures such as value at risk (VaR) and conditional value at risk (CVaR) have been used in many papers. Combined with different premium principles, the optimality results of optimal reinsurance are derived by minimizing VaR and CVaR of the insurer's total risk exposure. For instance, Cai and Tan (2007) determines explicitly the optimal retention level of a stop-loss reinsurance under the expectation premium principle. Tan et al. (2009) extends the study of Cai and Tan (2007) to other reinsurance premium principles associated with quota-share and stop-loss reinsurance. Motivated by Cai and Tan (2007), Cai et al. (2008) derives the optimal ceded loss functions among the class of increasing and convex ceded loss functions. Compared with Cai et al. (2008), Chi and Tan (2011) relaxes the constraints on the distribution of the aggregation loss and provide a simpler proof. Moreover, Chi and Tan (2011) considers a feasible class with constraints on the ceded and retained loss function, i.e., both the ceded and retained loss functions are increasing. See also, Bernard and Tian (2009), Cheung (2010), Tan et al. (2011), and references therein.

In terms of optimal reinsurance models proposed in these papers, a common assumption is that the distribution function of the total loss is known and satisfies some desirable properties. Then the tail risk measures can be analyzed regularly for a certain confidence level, and the reinsurance premiums can be calculated according to the premium principle. However, in practice, we may not have enough information to estimate the distribution of the total loss. For example, in catastrophe insurance, the loss data caused by the extreme event is scarce due to the low frequency of occurrence.

In the present paper, we assume that some incomplete information of the total loss is available, say its first two moments and support. More explicitly, let X be the total loss for an insurer, which belongs to the set $\mathcal{B} = \mathcal{B}(I; \mu, \sigma)$ of all nonnegative random variables with mean μ, standard deviation σ and support contained in the interval $I = [0, b]$, here $b = +\infty$ is allowed. Note that the partial knowledge is a reasonable assumption. This has been pointed out by several authors in actuarial and financial research, see e.g. Schepper and Heijnen (2007), Gerber and Smith (2008), De Schepper and Heijnen (2010), Wong and Zhang (2013), and references therein.

Following Cai and Tan (2007), the objective of this paper is to determine the optimal retention in a stop-loss reinsurance

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under expectation premium principle. In this paper, only partial information of the total loss rather than its distribution function is known. It is difficult to use VaR as a criterion in our case. Instead, we use an upper bound of VaR as our optimization criterion.

Actually, the optimization problem using VaR criterion for a stop-loss reinsurance involves two components in general: the evaluation of the VaR of the retained loss for the insurer and the calculation of pure reinsurance premium determined by certain principle, both require the knowledge of the distribution function of the risk X. When only incomplete information of X is available, the question arising here is whether we could find the optimal retention for a stop-loss reinsurance.

Inspired by distribution-free method, it is possible to derive stochastic bounds for certain risk in its moment space, which provides useful information in probabilistic modelling and has been widely adopted in actuarial literature. For example, Hürlimann (2001) calculates four plausible premium principles of risk X with known first two moments and bounded support. Given fixed few moments, Hürlimann (2002) yields the maximum value of VaR (CVaR) for risk X by construction extremal random variable with respect to (w.r.t.) the stochastic dominance order (stop-loss order). Assuming that the first two moments and support of X are known, Hürlimann (2005) uses the stop-loss ordered random variables to develop the analytical lower and upper bounds of X, and approximates pure premiums for excess of loss reinsurance with reinstatements. These papers have shown that the obtained approximations are accurate enough for practical purpose, especially when one agrees to calculate some risk measures, not based on the actual loss function, but based on stochastic bounds of the loss. For further reference, we refer readers to Hürlimann (2008a,b).

Consider a stop-loss reinsurance contract, the first part of this paper establishes an upper bound of the VaR of the total loss for the insurer. Furthermore, over the set $\mathcal{B}$, the VaR of the retained loss for the insurer is bounded by determining its maximum random variable w.r.t. the stochastic dominance order, and the reinsurance premium determined using expectation principle is bounded by constructing its maximum random variable w.r.t. the stop-loss order. The second part derives the optimal retention level as well as the sufficient and necessary conditions for the existence of the nontrivial optimal stop-loss reinsurance strategy by minimizing the obtained upper bound of the VaR.

The rest of the paper is structured as follows. Section 2 introduces the VaR based optimal stop-loss reinsurance model. Section 3 provides the distribution-free approximations and establishes an upper bound for the VaR of the total loss of the insurer. Section 4 derives the optimal retention and discusses the sufficient and necessary conditions for the existence of a nontrivial optimal stop-loss reinsurance strategy. The final Section 5 illustrates the results by numerical examples and compares them with the results yielded in Theorem 2.1 of Cai and Tan (2007).

2. VaR based optimal reinsurance model

In this section, we establish the framework of the VaR risk measure based optimal stop-loss reinsurance model, which have been described in detail in Cai and Tan (2007).

Let the total loss for an insurer be X, where $X \in \mathcal{B}$. We define X_I and X_R, respectively, as the retained loss and the ceded loss random variables under stop-loss reinsurance arrangement. Then X_I and X_R are related to X as follows:

$$X_I = \begin{cases} X, & X \le d \\ d, & X > d \end{cases} = X \wedge d \tag{1}$$

and

$$X_R = \begin{cases} 0, & X \le d \\ X - d, & X > d \end{cases} = (X - d)_+, \tag{2}$$

where $0 \le d \le b$ is known as the retention, $x \wedge y := \min\{x, y\}$, and $(x)_+ := \max\{x, 0\}$.

With the stop-loss reinsurance contract, the insurer caps the risk exposure at the retention, and transfers the part that exceeds the retention to the reinsurer. Note that $d = b$ denotes the special case where the insurer retains all loss, and $d = 0$ means that the insurer transfers all loss to the reinsurer. Consequently, the former case implies no reinsurance, and the latter case leads to full reinsurance.

In exchange of undertaking the risk, the insurer should pay a reinsurance premium to the reinsurer. Here, we assume that the reinsurance premium is determined by expectation principle and expressed as $\delta(d) = (1+\rho)\pi_X(d)$, where $\rho > 0$ is the safety loading factor and

$$\pi_X(d) = E(X_R) = E(X - d)_+ \tag{3}$$

is the stop-loss pure premium. In what follows, we denote $\bar{\rho} = (1+\rho)$ for simplicity.

Suppose that the total risk exposure of the insurer in the presence of reinsurance is T. The above analysis indicates that T can be expressed as the sum of two components: the retained loss and the incurred reinsurance premium; that is,

$$T = X_I + \delta(d). \tag{4}$$

To determine the optimal retention of stop-loss reinsurance by minimizing the proposed risk measure associated with T, we now introduce the definition of VaR.

The VaR of a random variable X at a confidence level $1-\alpha$ where $0 < \alpha < 1$ is defined as

$$VaR_\alpha(X) = \inf\{x : \Pr(X > x) \le \alpha\}. \tag{5}$$

It is equivalent to the $100(1-\alpha)$-th percentile of X. Hence,

$$VaR_\alpha(X) \le x \Leftrightarrow \bar{F}_X(x) \le \alpha, \tag{6}$$

where $\bar{F}_X(x) = 1 - F_X(x)$. In addition, if g is an increasing continuous function, then

$$VaR_\alpha(g(X)) = g(VaR_\alpha(X)). \tag{7}$$

Other properties of the VaR considered in this paper are its useful links with stochastic order, which will be presented in next section.

Analogously, we can define VaR for the insurer's retained loss X_I and the insurer's total loss T, i.e., $VaR_\alpha(d, X_I) = \inf\{x : \Pr(X_I > x) \le \alpha\}$ and $VaR_\alpha(d, T) = \inf\{x : \Pr(T > x) \le \alpha\}$. Here, we introduce an argument d to the VaR notations to emphasize that these risk measures are functions of the retention d. From (4) and (7), we have

$$VaR_\alpha(d, T) = VaR_\alpha(d, X_I) + \delta(d). \tag{8}$$

Building upon these, the optimal retentions by minimizing the corresponding VaR can be summarized as:

$$VaR_\alpha(d^*, T) = \min_{0 \le d \le b}\{VaR_\alpha(d, X_I) + \delta(d)\}. \tag{9}$$

In Cai and Tan (2007), the authors establish necessary and sufficient conditions for the existence of the optimal retention for (9), where the distribution function of the risk X plays a role in the resulting optimal solution. As previously mentioned, with only partial information of X, neither $VaR_\alpha(d, X_I)$ nor $\delta(d)$ can be derived analytically. Therefore, it is difficult to determine the optimal retention d^* in formula (9) for this case.

However, notice that in (9), we have that

$$VaR_\alpha(d, X_I) = VaR_\alpha(X) \wedge d, \qquad \delta(d) = \bar{\rho}\pi_X(d)$$

are two functionals of X, where the first equation holds due to (7). These two functionals preserve, respectively, the stochastic dominance order and the stop-loss order. Consequently, these orders exploiting results can be used to bound the functionals of X by determining their extremal values over the set $\mathcal{B}$, which will be explicitly introduced in next section.