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On minimizing drawdown risks of lifetime investments

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ABSTRACT

Drawdown measures the decline of portfolio value from its historic high-water mark. In this paper, we study a lifetime investment problem aiming at minimizing the risk of drawdown occurrences. Under the Black–Scholes framework, we examine two financial market models: a market with two risky assets, and a market with a risk-free asset and a risky asset. Closed-form optimal trading strategies are derived under both models by utilizing a decomposition technique on the associated Hamilton–Jacobi–Bellman (HJB) equation. We show that it is optimal to minimize the portfolio variance when the fund value is at its historic high-water mark. Moreover, when the fund value drops, the proportion of wealth invested in the asset with a higher instantaneous rate of return should be increased. We find that the instantaneous return rate of the minimum lifetime drawdown probability (MLDP) portfolio is never less than the return rate of the minimum variance (MV) portfolio. This supports the practical use of drawdown-based performance measures in which the role of volatility is replaced by drawdown.

1. Introduction

Drawdown, measuring the decline of portfolio value from its historic high-water mark, is a frequently quoted risk metric to evaluate the performance of portfolio managers in the fund management industry (see, e.g., Burghardt et al., 2003). Drawdown focuses primarily on extreme downward risks (as opposed to other standard risk measures such as volatility and Beta), making it particularly relevant for risk management purposes. Also, drawdown can easily be measured and interpreted by both portfolio managers and clients. A significant drawdown not only leads to large portfolio losses but may also trigger a long-term recession. Bailey and Lopez de Prado (2015) recently provided some justification to the so-called "triple penance rule", where the recovery period was shown to be on average three times as long as the time to produce a drawdown. Also, drawdown is considered a key determinant of sustainable investments as investors tend to overestimate their tolerance to risk. For instance, a sharp drop in portfolio's value is often accompanied by investors exercising their fund redemption options. Moreover, investors tend to assess their investment success by comparing their current portfolio value to the historical maximum value. This resulted in much hardship during the global financial crisis of 2008 when substantial drops in portfolio value were experienced across the board. Therefore, portfolio managers have strong incentives to adopt strategies with low drawdown risks (and more stable growth rate).

Portfolio optimization problems related to drawdown risks have long focused on maximizing the long-term (asymptotic) growth rate of a portfolio subject to a strict drawdown constraint. Grossman and Zhou (1993) pioneered this research topic by considering a market model with a risky asset and a risk-free asset in the Black–Scholes framework. This problem has been extended to a multi-asset framework and a general semimartingale framework by Cvitanic and Karatzas (1995) and Cherny and Obloj (2013), respectively. Klass and Nowicki (2005) later showed that the strategy proposed by Grossman and Zhou (1993) is not always optimal in a discrete-time setting. Moreover, the objective to maximize the long-term growth rate has been criticized because any strategy which coincides with the optimal strategy of Grossman and Zhou (1993) after any fixed time is optimal. Roche (2006) studied the infinite-horizon optimal consumption–investment problem for a power utility subject to the same drawdown constraint. Elie and Touzi (2008) later extended Roche (2006) to a general class of utility functions. Portfolio optimization problems with drawdown constraints are also considered in discrete-time settings (see, e.g., Chekhlov et al., 2005 and Alexander and Baptista, 2006).

In this paper, we consider the optimization problem of minimizing the probability that a significant drawdown occurs over a lifetime investment. Mathematically speaking, our problem

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is formulated as follows. On a filtered complete probability space $(\Omega, \mathcal{F}, \boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, we consider a $\boldsymbol{F}$-progressively measurable trading strategy $\pi = \{\pi_t\}_{t\geq 0}$. The associated fund value process is denoted by $W^\pi = \left\{W_t^\pi\right\}_{t\geq 0}$ with initial value $W_0 = w > 0$. We define the (floored) running maximum of the fund value at time t by

$$M_t^\pi = \max\Big\{\sup_{0\leq s\leq t} W_s^\pi, m\Big\},$$

with $m \geq w$. Note that the initial values w and m are fixed positive constants, and hence are independent of the trading strategy π. The ratios $(M_t^\pi - W_t^\pi)/M_t^\pi$ and W_t^π/M_t^π are respectively called the *relative drawdown level* and the *relative fund level* at time t. To quantify and measure the drawdown risk, for a fixed significance level $\alpha \in (0, 1)$, we define

$$\tau_\alpha^\pi = \inf\left\{t \geq 0 : M_t^\pi - W_t^\pi > \alpha M_t^\pi\right\},$$

to be the first time the relative drawdown of the fund value W^π exceeds the significance level $100\alpha\%$. Equivalently, the event $(\tau_\alpha^\pi > t)$ for some fixed $t > 0$ implies that the relative drawdown of the fund value in time period $[0, t]$ never exceeds α.

Our main objective is to solve for the optimal trading strategy $\pi^* = \left\{\pi_t^*\right\}_{t\geq 0}$ that minimizes the probability that a relative drawdown of size over α occurs before e_λ, the random time of death of a client with constant force of mortality $\lambda > 0$, i.e.,

$$\min_{\pi\in\Pi} \mathbb{P}\left\{\tau_\alpha^\pi < e_\lambda | W_0 = w, M_0 = m\right\}, \tag{1.1}$$

where Π is the set of admissible trading strategies defined as

$$\Pi = \Bigg\{\pi : \pi \text{ is } \boldsymbol{F}\text{-progressively measurable and } \int_0^t \pi_s^2 \mathrm{d}s < \infty \text{ for any } t \geq 0\Bigg\}. \tag{1.2}$$

Thus, e_λ is an $\mathcal{F}$-measurable exponentially distributed random variable with mean $1/\lambda > 0$, independent of the fund value process by assumption. For ease of notation, we denote the objective function in (1.1) as

$$\psi(w, m) = \min_{\pi\in\Pi} \mathbb{P}^{w,m}\left\{\tau_\alpha^\pi < e_\lambda\right\} = \min_{\pi\in\Pi} \mathbb{E}^{w,m}[e^{-\lambda\tau_\alpha^\pi}], \tag{1.3}$$

where the last equation is due to the independence of τ_α^π and e_λ. Here and henceforth, we write $\mathbb{E}^{w,m}[\,\cdot\,] = \mathbb{E}[\,\cdot\,|W_0 = w, M_0 = m]$.

The present work proposes to minimize the lifetime drawdown probability rather than impose a strict drawdown constraint, as is commonly done in the literature. This is because a strict drawdown constraint may not be attainable in some contexts (such as those discussed in Sections 2 and 3). As for other similar optimization problems (e.g., the minimum lifetime ruin probability (MLRP) of Young (2004), Bayraktar and Young (2007), Bayraktar and Zhang (2015) and references therein), we consider the drawdown probability over the lifetime of a client with a constant force of mortality. For the treatment of non-constant forces of mortality, one may adopt the approximative scheme of Moore and Young (2006). Finally, the solution of our resulting Hamilton–Jacobi–Bellman (HJB) equation does not possess a simple form, which makes its solution form difficult to guess. Instead, we decompose the HJB equation into two nonlinear equations of first order which are solved consecutively.

We point out that a recent paper by Angoshtari et al. (2015b) also studied the minimum drawdown probability problem but over an infinite-time horizon. By utilizing the results of Bäuerle and Bayraktar (2014), the authors found that the minimum infinite-time drawdown probability (MIDP) strategy coincides with the minimum infinite-time ruin probability (MIRP) strategy which consists in maximizing the ratio of the drift of the value process to its volatility squared. However, we point out that such a relationship does not hold for a random (or finite) maturity setting such as in (1.3) as the time-change arguments in Bäuerle and Bayraktar (2014) do not apply.

We will study the MLDP problem (1.3) by examining two different market models: a market with two risky assets and a market with a risk-free asset and a risky asset. We point out that several conclusions and implications of market model I are determinant to the subsequent analysis of market model II. Also, the following financial implications hold for both market models: (1) it is optimal to minimize the portfolio's variance when the fund value is at its historic high-water mark; (2) when the fund value drops, it is optimal to increase the proportion invested in the asset with a higher instantaneous rate of return (even though its volatility may also be higher). It follows that the instantaneous return rate of the MLDP strategy is never less than the return rate of the minimum variance (MV) strategy, which supports the practical use of drawdown-based performance measures.

The rest of the paper is organized as follows. The parallel Sections 2 and 3 are respectively devoted to the market models I and II. For each model, we provide a verification theorem, obtain closed-form expressions for the MLDP and its corresponding optimal trading strategy, as well as prove some properties of the optimal trading strategy. At the end of each section, we complement the analysis with some numerical examples.

2. Market model I

In this section, we study problem (1.3) under the market model consisting of two risky assets. We assume that the ith risky asset $(i = 1, 2)$ is governed by a geometric Brownian motion with dynamics

$$\mathrm{d}S_t^{(i)} = \mu_i S_t^{(i)} \mathrm{d}t + \sigma_i S_t^{(i)} \mathrm{d}B_t^{(i)}, \quad S_0^{(i)} > 0,$$

where $\mu_i \in \mathbb{R}, \sigma_i > 0$, and $\{B_t^{(i)}\}_{t\geq 0}$ is a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \boldsymbol{F}, \mathbb{P})$. In addition, $\{B_t^{(1)}\}_{t\geq 0}$ and $\{B_t^{(2)}\}_{t\geq 0}$ are assumed to be dependent with

$$\mathrm{d}B_t^{(1)}\mathrm{d}B_t^{(2)} = \rho \mathrm{d}t,$$

where $\rho \in (-1, 1)$ is the correlation coefficient. To avoid triviality, we exclude cases where the two assets are either perfectly positively or negatively correlated. Given a trading strategy $\pi \in \Pi$ defined in (1.2), where π_t represents *the fraction of wealth invested in Asset* 1 at time t, the evolution of the fund value process W^π is governed by

$$\begin{aligned} \mathrm{d}W_t^\pi &= \pi_t W_t^\pi \frac{\mathrm{d}S_t^{(1)}}{S_t^{(1)}} + (1 - \pi_t) W_t^\pi \frac{\mathrm{d}S_t^{(2)}}{S_t^{(2)}} \\ &= (\pi_t\mu_1 + (1 - \pi_t)\mu_2) W_t^\pi \mathrm{d}t + \pi_t W_t^\pi \sigma_1 \mathrm{d}B_t^{(1)} \\ &\quad + (1 - \pi_t) W_t^\pi \sigma_2 \mathrm{d}B_t^{(2)} \end{aligned} \tag{2.1}$$

with initial value $W_0 = w > 0$.

2.1. Verification theorem

We first prove a verification theorem for the MLDP. By a dimension reduction, the MLDP problem (1.3) will later be reduced to a one-dimensional stochastic control problem.

Let

$$D = \left\{(w, m) \in \mathbb{R}^2 : m(1 - \alpha) \leq w \leq m \text{ and } m > 0\right\},$$

and define a differential operator $\mathcal{L}^\beta$ $(\beta \in \mathbb{R})$ as

$$\mathcal{L}^\beta f = (\beta\mu_1 + (1 - \beta)\mu_2) x f_x + \frac{1}{2}\left(\beta^2\sigma_1^2 + (1 - \beta)^2\sigma_2^2 + 2\rho\beta(1 - \beta)\sigma_1\sigma_2\right) x^2 f_{xx} - \lambda f,$$