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Two maxentropic approaches to determine the probability density of compound risk losses

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HIGHLIGHTS

- We present two maxentropic methods to find a distribution of losses.
- The input consists of a few values of the Laplace transform estimated from the data.
- The density is reconstructed from a large sample of simulated data.
- The quality of the reconstruction is measured by a variety of statistical tests.
- The procedures will be extended to include errors in the data.

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ABSTRACT

Here we present an application of two maxentropic procedures to determine the probability density distribution of a compound random variable describing aggregate risk, using only a finite number of empirically determined fractional moments. The two methods that we use are the Standard method of Maximum Entropy (SME) and the method of Maximum Entropy in the Mean (MEM). We analyze the performance and robustness of these two procedures in several numerical examples, in which the frequency of losses is Poisson and the individual losses are lognormal random variables. We shall verify that the reconstructions obtained pass a variety of statistical quality criteria, and provide good estimations of VaR and TVaR, which are important measures for risk management purposes. As side product of the work, we obtain a rather accurate numerical description of the density of such compound random variable.

These approaches are also used to develop a procedure to determine the distribution of the individual losses from the knowledge of the total loss. Thus, if the only information available is the total loss, and the nature of the frequency of losses is known, the method of maximum entropy provides an efficient method to determine the individual losses as well.

1. Introduction

Both in the insurance and the banking industries it is important to know how to compute the density of a compound random variable describing an accumulated random number of losses. In the banking industry this is the first step towards the implementation of the advanced measurement approach to determine regulatory capital, and in the insurance industry it is the first step to determine insurance premia.

To be specific, in this work we shall suppose that the frequency of losses in a given period of time is described by a compound random variable of the type $S = \sum_{j\geq 0}^{N} X_j$, where N is a Poisson random variable of intensity ℓ, and $\{X_j, \text{ for } j = 1, \ldots, N\}$ denote the individual losses which are independent and identically distributed. This type of problems has been studied for a long time, and a variety of techniques exist for its solution, see for example Panjer (2006), but techniques like those proposed here are not yet widely used.

From an abstract point of view, our implementation of the maxentropic methods fall within the techniques to invert Laplace transform from a few values of the parameter along the real axis. We have actually tried that in a situation in which the Laplace transform could be determined analytically and its values along the real axis are known. In Gzyl et al. (2013) the authors applied

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the SME approach (along with other methodologies) to find the probability density of a compound random variable, in which the frequency is Poisson and the individual losses were $\Gamma(a, b)$. In this case, the compound density f_S may be approximated to any desired degree and different methods of reconstruction can be compared.

But when the individual losses follow a lognormal distribution, Laplace transform techniques are hard to implement because, to begin with, the Laplace transform of a lognormal density is unknown. To wit, consider the effort to compute it approximately carried out by Leipnik (1991). Thus, the example that we consider in this paper is such that a direct analytical solution is not known, and the standard numerical procedures to determine them are hard to implement. Here we propose a direct procedure which we think that is better to solve the problem of determining the density of the total sum of lognormal individual losses when the frequency is of the Poisson type through the use of a simulated sample. We will choose a large enough sample to provide with the best approximation possible. This is one approximation to the problem and further extensions include. As this may happen in many cases of practical interest, our chosen example is typical in this regard.

Besides that, the lognormal distribution is frequently used to model individual claims in various classes of insurance business and in risk theory to model losses caused by different risk events. The fact that it has a heavy tail is important, because it allows us consider the possibility of describing very large claims, which correspond to losses that threaten the solvency of an insurance company or a bank. This has important implications for the determination of premiums, risk reserves and reinsurance (Crow and Shimizu, 1988).

The starting point for us will be the Laplace transform of $S(N)$ (or S for short)

$$E[e^{-\alpha_i S}] = \psi(\alpha_i) = \int_0^\infty e^{-\alpha_i s} dF_S(s), \quad i = 1, \ldots, K \tag{1}$$

which will be calculated numerically from simulated data. To do that, set $Y = e^{-S}$ and we transform the Laplace inversion problem into the problem of inferring the density $f_Y(y)$ from fractional moments. For that, we think of the previous identity as follows

$$\psi(\alpha_i) = E[Y^\alpha] = \int_0^1 y^{\alpha_i} dF_Y(y), \quad i = 1, \ldots, K. \tag{2}$$

Note now that, since the distribution F_S of S has a point mass $e^{-\ell}$ at $S = 0$, in order to relate the $\psi(\alpha)$ to the density $f_Y(y)$ of Y, we have to condition out the mass at $\{Y = 1\}$ (or the mass of F_S at $\{S = 0\}$). For that we consider the conditional version

$$\begin{aligned} E[e^{-\alpha_i S} \mid S > 0] &= \int_0^1 y^{\alpha_i} f_Y(y) dy \\ &= \frac{\psi(\alpha_k) - e^{-\ell}}{1 - e^{-\ell}} := \mu(\alpha_i), \quad i = 1, \ldots, K \end{aligned} \tag{3}$$

which defines the $\mu(\alpha_i)$ that will be the input for the maxentropic methods. Once f_Y has been determined, in order to recover f_S we have to apply the change of variables $y = e^{-s}$ to obtain $f_S(s) = e^{-s} f_Y(e^{-s})$.

The two versions of the maximum entropy method presented in this paper will be used to determine the distribution of the individual losses from the knowledge of the total severity. Also, in case we only have a historical record of the total losses and a model for the frequency of losses is available (and in our case it is), it is possible to decompound (or to disaggregate) the distribution of losses and obtain the distribution of individual losses. This could be useful for a risk manager that may want to know the distribution of the individual losses in order to apply any particular corrective loss prevention policy.

The remainder of the paper is organized as follows. We recall briefly the basic details of the SME and MEM methods in Section 2. In Section 3, we show the results of the implementation of the SME and MEM approaches to determine the distribution of total losses. At this point, we mention the SME and the MEM methods have been applied successfully in a large variety of problems, see Kapur (1989) for the standard formulation and examples in many fields. As far as applications in risk and insurance, the reader may want to check with Berliner (1984), Martin-Löf (1986) or Brocket (1991) and the several comments by the discussants. Consider as well Li (2010) and Haberman et al. (2011). Actually, one may say that the germ of the idea for the method of maximum entropy appears in the work by Esscher (1932), but the maximum entropy method as a variational method to determine probability densities seems to have been first proposed by Jaynes (1957). See also Gzyl and Velásquez (2011) for details and references, specially to the method of maximum entropy in the mean, and for a review on the applications of the standard method of maximum entropy to finance consider Zhou et al. (2013).

Section 4 is devoted to the computation of two of the most commonly used risk measures, namely the *VaR* and the *TVaR* using the maxentropic density as loss probability density. This could be interesting for risk managers who may consider insuring operational risk losses to decrease the capital charges. Section 5 is devoted to the complementary problem of decompounding, by means of which the distribution of individual losses is obtained from the total loss. In Section 6, we present some concluding remarks. Finally, in the Appendix we provide a quick overview of the statistical tests and graphical tools used to verify the robustness and the quality of the results.

2. The maxentropic approaches

Below we review the basis of the SME and MEM methods used to solve the problem of finding the density of the total severity from the knowledge of a small number of fractional moments. Both maxentropic procedures yield similar and good quality results. The procedure of maximum entropy in the mean is more general and contains the standard maximum entropy method as particular case, and it is useful to know about its existence.

2.1. The standard method of maximum entropy (SME)

This is a variational procedure to solve the (inverse) problem consisting of finding a probability density $f_Y(y)$ (on $[0, 1]$ in this case), satisfying the following integral constraints:

$$\int_0^1 y^{\alpha_k} f_Y(y) dy = \mu_Y(\alpha_k) \quad \text{for } k = 0, 1, \ldots, K. \tag{4}$$

We set $\alpha_0 = 0$ and $\mu_0 = 1$ to take care of the natural normalization requirement on $f_Y(y)$. The intuition is rather simple: The class of probability densities satisfying (4) is convex. One can pick up a point in that class one by maximizing (or minimizing) a concave (convex) functional (an "entropy") that achieves a maximum (minimum) in that class. That extremal point is the "maxentropic" solution to the problem. It actually takes a standard computation to see that, when the problem has a solution, it is of the type

$$f_K^*(y) = \exp\left(-\sum_{k=0}^{K} \lambda_k^* y^{\alpha_k}\right) \tag{5}$$

in which the number of moments K appears explicitly. It is usually customary to write $e^{-\lambda_0^*} = Z(\boldsymbol{\lambda}^*)^{-1}$, where $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots,$