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On a partial integrodifferential equation of Seal's type

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ABSTRACT

In this paper we generalize a partial integrodifferential equation satisfied by the finite time ruin probability in the classical Poisson risk model. The generalization also includes the bivariate distribution function of the time of and the deficit at ruin. We solve the partial integrodifferential equation by Laplace transforms with the help of Lagrange's implicit function theorem. The assumption of mixed Erlang claim sizes is then shown to result in tractable computational formulas for the finite time ruin probability as well as the bivariate distribution function of the time of and the deficit at ruin. A more general partial integrodifferential equation is then briefly considered.

1. Introduction

In the classical Poisson risk model, the insurer's surplus $\{U_t; t \geq 0\}$ is defined by $U_t = u+ct-S_t$, $t \geq 0$, where $u \geq 0$ is the insurer's initial surplus, c is the rate of premium income per unit time, and S_t is the aggregate claims up to time t. Then $S_t = \sum_{i=1}^{N_t} Y_i$ (with $S_t = 0$ if $N_t = 0$), where $\{N_t; t \geq 0\}$ is a Poisson process with Poisson rate λ, and $\{Y_i; i = 1, 2, \ldots\}$ a sequence of independent and identically distributed (i.i.d.) positive random variables independent of $\{N_t; t \geq 0\}$. The random variable Y_i represents the amount of the ith claim, and has distribution function (df) $P(y) = 1 - \overline{P}(y) = \Pr(Y \leq y)$, $y \geq 0$, density function $p(y) = P'(y)$, mean $E(Y) = \int_0^\infty yp(y)dy$, and Laplace transform (LT) $\tilde{p}(s) = \int_0^\infty e^{-sy}p(y)dy$. The positive loading condition is assumed, whereby $c = (1+\theta)\lambda E(Y)$ with $\theta > 0$. The aggregate claims S_t has mass point $\Pr(S_t = 0) = e^{-\lambda t}$ at 0, and density $f(y, t)$ for $y > 0$ given by

$$f(y,t) = \sum_{n=1}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} p^{*n}(y), \tag{1}$$

where $p^{*n}(y)$ is the density function of $Y_1 + Y_2 + \cdots + Y_n$ for $n = 1, 2, \ldots$. Then the Laplace transform of $f(y, t)$ is, from (1),

$$\tilde{f}(s,t) = \int_0^\infty e^{-sy} f(y,t)dy = e^{\lambda t\{\tilde{p}(s)-1\}} - e^{-\lambda t}, \tag{2}$$

and the df $F(y, t) = 1 - \overline{F}(y, t) = \Pr(S_t \leq y)$ satisfies

$$F(y,t) = e^{-\lambda t} + \int_0^y f(x,t)dx, \quad y \geq 0. \tag{3}$$

Of central importance is the time of ruin $T = \inf\{t; U_t < 0\}$, with $T = \infty$ if $U_t \geq 0$, $t \geq 0$. The infinite time ruin probability is $\psi(u) = \Pr(T < \infty | U_0 = u)$.

The finite time survival probability (e.g. Asmussen and Albrecher, 2010, Chapter V; Takacs, 1962, pp. 55–56; Seal, 1974; Seal, 1978; or Panjer and Willmot, 1992, Section 11.7) is defined to be $\phi(u, t) = \Pr(U_s \geq 0,\ 0 \leq s \leq t | U_0 = u)$. It is well known that $\phi(u, t)$ satisfies the partial integrodifferential equation

$$\begin{aligned} \frac{\partial}{\partial t}\phi(u,t) &= c\frac{\partial}{\partial u}\phi(u,t) - \lambda\phi(u,t) \\ &\quad + \lambda\int_0^u \phi(u-x,t)p(x)dx, \end{aligned} \tag{4}$$

and that the solution to (4) may be expressed as

$$\begin{aligned} \phi(u,t) &= F(u+ct,t) - c\int_0^t \phi(0,x) \\ &\quad \times f(u+c(t-x),t-x)\,dx, \quad u \geq 0, \end{aligned} \tag{5}$$

with $f(y, t)$ given by (1), $F(y, t)$ given by (3), and

$$\phi(0,t) = \frac{1}{ct}\int_0^{ct} F(y,t)dy. \tag{6}$$

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The deficit at ruin is $|U_T|$ (e.g. Asmussen and Albrecher, 2010), and the joint df of the time of ruin and the deficit at ruin is $G(u,t,y) = \Pr(T \le t, |U_T| \le y | U_0 = u)$. The function $G(u,t,y)$ satisfies the partial integrodifferential equation

$$\frac{\partial}{\partial t}G(u,t,y) = c\,\frac{\partial}{\partial u}G(u,t,y) - \lambda G(u,t,y) + \lambda \int_0^u G(u-x,t,y)p(x)dx + \lambda\left\{\overline{P}(u) - \overline{P}(u+y)\right\}. \tag{7}$$

To see (7), note that standard properties of the Poisson process and the method of infinitesimals yield (for a small time interval of length h)

$$G(u,t+h,y) = (1-\lambda h)G(u+ch,t,y) + \lambda h\left\{\overline{P}(u+ch) - \overline{P}(u+ch+y)\right\} + \lambda h \int_0^{u+ch} G(u+ch-x,t,y)p(x)dx + o(h),$$

from which (7) follows by rearrangement and letting $h \to 0$. We remark that Dickson (2012) has analyzed the joint density function of the time and the deficit using a different approach. The approach used in this paper is different, resulting in the joint distribution function.

Motivated by the form of (4) and (7), we consider the partial integrodifferential equation for a function $h(u,t)$ given by

$$\frac{\partial}{\partial t}h(u,t) = c\,\frac{\partial}{\partial u}h(u,t) - \lambda h(u,t) + \lambda \int_0^u h(u-x,t)p(x)dx + \tau(u). \tag{8}$$

Clearly, (4) is the special case of (8) with $\tau(u) = 0$, and (7) is the special case with $\tau(u) = \lambda\{\overline{P}(u) - \overline{P}(u+y)\}$. Thus, (8) unifies and generalizes these results.

In Section 2 we solve (8) by Laplace transforms, and express the solution in terms of the boundary function $h(u,0)$, which is assumed to be known (for example, $\phi(u,0) = 1$ and $G(u,0,y) = 0$ based on physical properties of the problem at hand).

In Section 3 we demonstrate that all quantities needed for the solution of $\phi(u,t)$ and $G(u,t,y)$ are readily obtainable without numerical integration for the important situation when $p(y)$ is the mixed Erlang density (e.g. Klugman et al., 2013, Chapter 3, and references therein). In particular, the resulting expression for the finite time ruin probability in the case of mixed Erlang claims appears to be simpler than that given by Dickson and Willmot (2005), which was obtained using Gerber–Shiu based arguments. Applications to the exponential and ordinary Erlang claim size distributions are also considered.

A more general equation than (8) is then considered in Section 4, where the Laplace transform approach to solution is seen to still be applicable.

2. The general solution

In order to solve (8), we will employ Laplace transforms. Thus, define

$$\tilde{\tau}(s) = \int_0^\infty e^{-su}\tau(u)du, \tag{9}$$

and

$$\tilde{h}_1(s,t) = \int_0^\infty e^{-su}h(u,t)du. \tag{10}$$

Therefore, taking Laplace transforms of (8), or equivalently, multiplying (8) by e^{-su} and integrating with respect to u from 0 to ∞, it follows from (9) and (10) that

$$\frac{\partial}{\partial t}\tilde{h}_1(s,t) = c\left\{s\tilde{h}_1(s,t) - h(0,t)\right\} - \lambda\tilde{h}_1(s,t) + \lambda\tilde{h}_1(s,t)\tilde{p}(s) + \tilde{\tau}(s). \tag{11}$$

Next, define the bivariate LT of $h(u,t)$ to be

$$\tilde{h}(s,z) = \int_0^\infty\int_0^\infty e^{-su-zt}h(u,t)dudt = \int_0^\infty e^{-zt}\tilde{h}_1(s,t)dt, \tag{12}$$

and the LT of $h(0,t)$ to be

$$\tilde{h}_0(z) = \int_0^\infty e^{-zt}h(0,t)dt. \tag{13}$$

Again, taking Laplace transforms of (11) with respect to t and using (12) and (13) results in

$$z\tilde{h}(s,z) - \tilde{h}_1(s,0) = cs\tilde{h}(s,z) - c\tilde{h}_0(z) - \lambda\tilde{h}(s,z) + \lambda\tilde{p}(s)\tilde{h}(s,z) + \frac{\tilde{\tau}(s)}{z},$$

which may be expressed as

$$\left\{z - cs + \lambda\left[1-\tilde{p}(s)\right]\right\}\tilde{h}(s,z) = \tilde{h}_1(s,0) + \frac{\tilde{\tau}(s)}{z} - c\tilde{h}_0(z). \tag{14}$$

Before proceeding to the general solution to $h(u,t)$ with bivariate LT $\tilde{h}(s,z)$ given by (12), we recall that we assume that $h(u,0)$ and hence $\tilde{h}_1(s,0)$ is known, and it is of interest to identify $h(0,t)$ with LT $\tilde{h}_0(z)$ given by (13). Then, equating the coefficient of $\tilde{h}(s,z)$ in (14) to 0 results in

$$z - cs + \lambda\left\{1-\tilde{p}(s)\right\} = 0. \tag{15}$$

When viewed as a function of s for fixed z, (15) is the classical Lundberg's fundamental equation which is of central importance in Gerber–Shiu analysis (Gerber and Shiu, 1998). In particular, it is well known that there is a unique root $r(z)$ to (15) with positive real part, and (any analytic function of) $r(z)$ may be obtained via Lagrange's implicit function theorem. See Benes (1957) for details. We are now in a position to identify $\tilde{h}_0(z)$ and $h(0,t)$, which is done in the following theorem.

Theorem 1. *The LT (13) satisfies*

$$\tilde{h}_0(z) = \frac{1}{c}\tilde{h}_1\left\{r(z),0\right\} + \frac{1}{c}\,\frac{\tilde{\tau}\{r(z)\}}{z}, \tag{16}$$

where $s = r(z)$ is the unique root of (15) with positive real part. Moreover, $h(0,t)$ is given explicitly by

$$h(0,t) = e^{-\lambda t}h(ct,0) + \frac{1}{c}\sum_{n=1}^\infty \frac{\lambda^n t^{n-1}e^{-\lambda t}}{n!}\int_0^{ct} xp^{*n}(ct-x)h(x,0)dx + \int_0^t\left\{e^{-\lambda v}\tau(cv) + \frac{1}{c}\sum_{n=1}^\infty\frac{\lambda^n v^{n-1}e^{-\lambda v}}{n!} \times \int_0^{cv} xp^{*n}(cv-x)\tau(x)dx\right\}dv. \tag{17}$$