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A reinsurance game between two insurance companies with nonlinear risk processes

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ABSTRACT

In this paper, we consider a stochastic differential reinsurance game between two insurance companies with nonlinear (quadratic) risk control processes. We assume that the goal of each insurance company is to maximize the exponential utility of the difference between its terminal surplus and that of its competitor at a fixed terminal time T. First, we give an explicit partition (including nine subsets) of time interval $[0, T]$. Further, on every subset, an explicit Nash equilibrium strategy is derived by solving a pair of Hamilton–Jacobi–Bellman equations. Finally, for some special cases, we analyze the impact of time t and quadratic control parameter on the Nash equilibrium strategy and obtain some simple partition of $[0, T]$. Based on these results, we apply some numerical analysis of the time t, quadratic control parameter and competition sensitivity parameter on the Nash equilibrium strategy and the value function.

1. Introduction

The optimal reinsurance problem is a classical issue in actuarial risk theory since reinsurance is a very useful tool for an insurance company (insurer) to transfer its risk exposures to a reinsurer. Literature on optimal reinsurance can be found in Cai and Tan (2007), Cai et al. (2008), Cheung (2010), Cai and Wei (2012), Cui et al. (2013) and Chi and Meng (2014) for a static, single-period risk model, and Choulli et al. (2003), Meng and Siu (2011), Li et al. (2012), Asimit et al. (2013) and Chen and Yam (2013) for a continuous time model and references therein.

Recently, stochastic differential games on reinsurance/investment have been studied extensively. For maximizing/minimizing a payoff function depending on the difference of two insurance companies' surplus processes, Zeng (2010) and Taksar and Zeng (2011) studied a zero-sum stochastic differential game between two insurers by applying proportional and non-proportional reinsurance, respectively. Bensoussan et al. (2014) studied the relative performance of two insurance companies under a non-zero sum stochastic differential game framework.

It appears that the vast literature mainly focuses on linear risk processes. However, Guo et al. (2004) pointed out that the oversimple linear relation between risk and return stemming from proportional reinsurance violates probably the fact that excessive risk exposure may not be a recipe for a high return. With the consideration of internal competition factors of reinsurance markets, Guo et al. (2004) proposed a quadratic nonlinear risk model with absence of investment, that is, the risk process follows

$$R(t) = x + \int_0^t \left(\widetilde{\mu}\pi(s) - a(\pi(s)-p)^2 - \widetilde{\delta}\right)\mathrm{d}s + \int_0^t \sigma\pi(s)\mathrm{d}W(s)$$
$$=: x + \int_0^t (\mu\pi(s) - a\pi^2(s) - \delta)\mathrm{d}s + \int_0^t \sigma\pi(s)\mathrm{d}W(s),$$

where $\tilde{\mu}, a, \sigma, \tilde{\delta}, p$ are given constants; $W(\cdot)$ is one-dimensional standard Brownian motion; $\pi(t)$ is an admissible control. Specially, p is the preferred reinsurance level imposed by a risk-averse reinsurer, a is the extra rate of charge for the deviation from the preferred level and $\int_0^t a(\pi(s)-p)^2\mathrm{d}s$ is an additional amount of service charge on firms seeking services beyond/below the target level p before the time t. In Meng et al. (2013), introduction of a quadratic control term in a nonlinear risk process is explained as originating from the use of the variance premium principle. Specifically, with the variance premium principle, the diffusion approximation to a compound Poisson risk process with proportional

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reinsurance and debt can be written as:

$$R(t) = x + \int_0^t (2\widetilde{\mu}\pi(s) - \widetilde{\mu}^2\pi^2(s) - \delta)\mathrm{d}s + \int_0^t \sigma\pi(s)\mathrm{d}W(s).$$

Thus, by expanding $2\widetilde{\mu}$ and $\widetilde{\mu}^2$ to any parameters $\mu > 0$ and $a > 0$, respectively, we obtain the corresponding nonlinear risk process, which explains intuitively the emergence of quadratic control term. Literature on the variance premium principle includes Zhou and Yuen (2012), Jin et al. (2013), and Liang and Yuen (2014). Besides, Guo (2002) described the model $R(t)$ as a workforce control problem and interpreted the factor $\pi(t)$ and a as the number of hires at time t and a friction coefficient (reflecting the counter-productivity phenomenon of over-hiring), respectively.

The above nonlinear risk model fully reflects the reinsurance balance between the insurance company and the reinsurance company. In addition, there are some dependent and competitive relations between two different insurance companies. Based on the above literature and compared with Bensoussan et al. (2014), we will analyze the competitions between two insurance companies with nonlinear risk processes, which may be more meaningful in reality. With the exponential utility function, we consider the problem of optimal reinsurance when one insurer takes into account its relative performance against the other insurer, i.e., reinsurance game problem.

The rest of the paper is organized as follows. Section 2 presents a modeling framework and formulates a reinsurance game problem. Section 3 formulates the game problem via a pair of Hamilton–Jacobi–Bellman (HJB) equations and gives an explicit solution for the Nash equilibrium strategy. In Section 4, we study some special cases and perform some numerical analysis on the Nash equilibrium strategy and the value function. The final section concludes the paper.

2. Model formulation

Inspired by Guo (2002) and Guo et al. (2004), we consider a market with two competing reinsurance controlled processes $\overline{X}_i^{\Pi_i}(t)$, for $i = 1, 2$, allowing insurers to invest their surpluses in a risk-free asset, with $\overline{X}_i^{\Pi_i}(t)$ being given by

$$\mathrm{d}\overline{X}_i^{\Pi_i}(t) = \left[\mu_i\pi_i(t) - a_i\pi_i^2(t) - \delta_i + r\overline{X}_i^{\Pi_i}(t)\right]\mathrm{d}t + \sigma_i\pi_i(t)\mathrm{d}W_i(t). \tag{2.1}$$

Here $W_i(\cdot)$ is one-dimensional standard Brownian motion on a complete, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and $\mathrm{d}\langle W_1(t), W_2(t)\rangle = \rho\mathrm{d}t$, $\rho \in [-1, 1]$ is the correlation coefficient; $\mu_i, a_i, \delta_i, r, \sigma_i > 0$ are given constants, μ_i is the expected profit rate, σ_i is the volatility rate, δ_i is the debt rate, r is the interest rate, a_i is the additional rate of charge for the deviation from the preferred level; $\pi_i(t) \in [\alpha, \beta]$ is a control process, and $0 \leq \alpha < \beta < +\infty$. Model (2.1) can be explained fully that a risk-averse reinsurer has a preferred risk level and wishes to impose additional service charges on the insurance company seeking services beyond/below the target level.

We remark that if $\alpha = 0$ and $\beta = 1$, model (2.1) is a risk process with usual reinsurance. For narrational convenience, we still call (2.1) as a reinsurance controlled process under general case $0 \leq \alpha < \beta < +\infty$. Note that when $\pi_i(t) > 1$, one can interpret the control as acquiring new business, see Bäuerle (2005). The coefficient a_i of the quadratic control $\pi_i^2(t)$ can be explained as internal competition factor (see Guo et al., 2004) or stemming from the use of the variance premium principle (see Meng et al., 2013).

The control $\Pi_i = \{\pi_i(t)\}_{t\geq 0}$ is said to be admissible if Π_i satisfy: (i) $\{\pi_i(t)\}_{t\geq 0}$ is the $\mathcal{F}$-progressively measurable process; (ii) $\pi_i(t) \in [\alpha, \beta]$. Write $\mathcal{H}$ for the space of all admissible strategies.

Similar to Espinosa and Touzi (2013) and Bensoussan et al. (2014), we assume that the prime objective of insurer i is to maximize the expected utility of his performance at the terminal time $T \in (0, \infty)$, relative to its competitor, i.e.,

$$\begin{aligned}\sup_{\Pi_i\in\mathcal{H}} \mathbb{E}_{t,x_i}&\left[U_i\left(\overline{X}_k^{\Pi_i}(T) - \kappa_i\overline{X}_j^{\Pi_j}(T)\right)\right]\\ &= \sup_{\Pi_i\in\mathcal{H}} \mathbb{E}\left[U_i\left(\overline{X}_i^{\Pi_i}(T) - \kappa_i\overline{X}_j^{\Pi_j}(T)\right)\middle|\overline{X}_i^{\Pi_i}(t) - \kappa_i\overline{X}_j^{\Pi_j}(t) = x_i\right],\end{aligned} \tag{2.2}$$

where $U_i(\cdot)$ is a utility function, $i, j \in \{1, 2\}$, $i \neq j$, $\kappa_i \in [0, 1]$ and $0 \leq t \leq T$.

For simplification, we assume that each insurer has an exponential utility function with

$$U_i(x) = -\frac{1}{\eta_i}\exp(-\eta_i x), \quad \eta_i > 0,\ i = 1, 2,$$

which has constant absolute risk aversion (CARA) parameter η_k.

Obviously, problem (2.2) is equivalent to the following game problem.

Game problem: Find a Nash equilibrium strategy $(\Pi_1^*, \Pi_2^*) \in \mathcal{H}\times\mathcal{H}$ such that

$$\begin{aligned}&\mathbb{E}_{t,x_1}\left[U_1\left(\overline{X}_1^{\Pi_1}(T) - \kappa_1\overline{X}_2^{\Pi_2^*}(T)\right)\right] \leq \mathbb{E}_{t,x_1}\left[U_1\left(\overline{X}_1^{\Pi_1^*}(T) - \kappa_1\overline{X}_2^{\Pi_2^*}(T)\right)\right],\\ &\mathbb{E}_{t,x_2}\left[U_2\left(\overline{X}_2^{\Pi_2}(T) - \kappa_2\overline{X}_1^{\Pi_1^*}(T)\right)\right] \leq \mathbb{E}_{t,x_2}\left[U_2\left(\overline{X}_2^{\Pi_2^*}(T) - \kappa_2\overline{X}_1^{\Pi_1^*}(T)\right)\right].\end{aligned} \tag{2.3}$$

3. An explicit solution

In this section, we aim to search for the Nash equilibrium reinsurance strategy $(\Pi_1^*, \Pi_2^*) \in \mathcal{H}\times\mathcal{H}$. Let $X_i^{\Pi_i,\Pi_j}(t) = \overline{X}_i^{\Pi_i}(t) - \kappa_i\overline{X}_j^{\Pi_j}(t)$ for $i, j \in \{1, 2\}$, $i \neq j$. Thus

$$\begin{aligned}\mathrm{d}X_i^{\Pi_i,\Pi_j}(t) = &\Big[rX_i^{\Pi_i,\Pi_j}(t) + \mu_i\pi_i(t) - a_i\pi_i^2(t) - \delta_i - \kappa_i\mu_j\pi_j(t) + \kappa_i a_j\pi_j^2(t) + \kappa_i\delta_j\Big]\mathrm{d}t\\ &+ \sigma_i\pi_i(t)\mathrm{d}W_i(t) - \kappa_i\sigma_j\pi_j(t)\mathrm{d}W_j(t).\end{aligned} \tag{3.1}$$

Define, for $0 \leq t \leq T$,

$$\begin{aligned}V_i^{\Pi_j}(t, x_i) &\triangleq \sup_{\Pi_i\in\mathcal{H}} \mathbb{E}_{t,x_i}U_i\left(X_i^{\Pi_i,\Pi_j}(T)\right)\\ &= \sup_{\Pi_i\in\mathcal{H}} \mathbb{E}_{t,x_i}\left[U_i\left(\overline{X}_i^{\Pi_i}(T) - \kappa_i\overline{X}_j^{\Pi_j}(T)\right)\right].\end{aligned} \tag{3.2}$$

Define the operator

$$\begin{aligned}\mathcal{L}^{\pi_i,\pi_j}W^i(t, x_i) = &\left[rx_i + \mu_i\pi_i - a_i\pi_i^2 - \delta_i - \kappa_i\mu_j\pi_j + \kappa_i a_j\pi_j^2 + \kappa_i\delta_j\right]W_x^i(t, x_i)\\ &+ \frac{1}{2}\left[\sigma_i^2\pi_i^2 - 2\sigma_i\pi_i\kappa_i\sigma_j\pi_j\rho + \kappa_i^2\sigma_j^2\pi_j^2\right] \times W_{xx}^i(t, x_i).\end{aligned} \tag{3.3}$$

By the standard arguments (see Fleming and Soner, 1993), we know that, for any admissible Π_j with $\pi_j(t) = \pi_j$, $V_i^{\Pi_j}(t, x_i)$ satisfies the following Hamilton–Jacobi–Bellman (HJB) equation

$$W_t^i(t, x_i) + \sup_{\pi_i\in[\alpha,\beta]} \mathcal{L}^{\pi_i,\pi_j}W^i(t, x_i) = 0. \tag{3.4}$$