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Minimal representation of insurance prices

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ABSTRACT

This paper prices insurance contracts by employing law invariant, coherent risk measures from mathematical finance. We demonstrate that the corresponding premium principle enjoys a minimal representation. Uniqueness – in a sense specified in the paper – of this premium principle is derived from this initial result. The representations are derived from a result by Kusuoka, which is usually given for nonatomic probability spaces. We extend this setting to premium principles for spaces with atoms, as this is of particular importance for insurance.

Further, stochastic order relations are employed to identify the minimal representation. It is shown that the premium principles in the minimal representation are extremal with respect to the order relations. The tools are finally employed to explicitly provide the minimal representation for premium principles, which are important in actuarial practice.

1. Introduction

Convex principles for insurance contracts are formulated in Deprez and Gerber (1985). They constitute an entire family of premium principles (cf. Young, 2006), which can be employed to price insurance contracts. More than ten years later the axioms contained there became popular as coherent risk measures, which have been introduced in mathematical finance by Artzner et al. (1999) to quantify the risk related to a financial exposure.

Kusuoka (2001) elaborates a comprehensive, explicit representation of these premium principles. The basic, elementary ingredient is well known in insurance as *Conditional Tail Expectation* (CTE, sometimes also *Conditional*, or *Average Value-at-Risk*). The Conditional Tail Expectation is employed in two ways in insurance: to price *individual* insurance contracts and to evaluate the risk related to an entire *portfolio* of contracts (for example by the US and Canadian insurance supervisory authorities, cf. Ko et al., 2009). Kusuoka's representation of a premium principle is a supremum of convex combinations of CTEs. In this way the CTE thus can be interpreted as an extremal point of a Choquet integral.

Choquet integrals of CTEs are also known as distortion risk measures, or Wang premium principle in insurance (cf. Wang, 2000 or Denneberg, 1990 for an early discussion of the concept). It is a main characteristic of these premium principles that they overvalue high risks, which are unfavorable for the insurer, while assigning less weight to negligible risks in exchange. Wang premiums constitute a convex premium principle, for which Pichler (2013a) elaborates the corresponding convex conjugate.

It is known that Kusuoka representations of premiums or risk measures are not unique (cf., e.g., Pflug and Römisch, 2007). However, it is demonstrated in Shapiro (2013) for the special case of distortion premiums that the corresponding Kusuoka representation allows a unique, minimal representation. This raises the question of in some sense minimality of representations of a general premium principle. It was also posed the question whether such a minimal representation is unique in general? In this paper we give a positive answer to this question and show how such minimal representations can be derived in a constructive way. Further characterizations derived involve stochastic dominance relations of first and second order.

We finally provide minimal representations of two particular premium principles in the concluding section. These premium principles are natural extensions of the Conditional Tail Expectation, but they assign higher weights to the tails. As a special case we provide a new closed form representation of the Dutch premium principle, which is a simple (and for this reason very useful) premium principle allowing a compelling natural interpretation. Its Kusuoka representation turns out to be a mixture of the net premium principle and the Conditional Tail Expectation with a specified weight.

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Kusuoka has elaborated his original results on probability spaces without atoms. As an extension we discuss Kusuoka representations for probability spaces which are not nonatomic. This is of particular interest for insurance contracts with a finite number of possible payouts in case of the insurance event, as the underlying samples are typically modeled as atoms with positive probability. Many life insurance contracts are notably of this specific form, where the sum insured is paid in case of death, say, or nothing. The survival of an insured person within one year is naturally given as an atomic insurance event with strictly positive measure, the survival probability. Our characterization of law invariant premium principles provides a necessary condition for existence of a Kusuoka representation on an atomic probability space. It follows from our results that the probabilities of the atoms have to be of a very special form in order to constitute a version independent premium principle, which are beyond the canonical embedding in the standard probability space (the probabilities of the atoms have to be equally weighted). This is not the case in actuarial practice, and it cannot be guaranteed when employing an empirical measure to determine the weights of the atoms. For this we conclude that additional premium principles are essentially not available in the case of a space with atoms, so that one has to use the traditional premium principles.

Kusuoka (2001) formulates his original results in $L_\infty(\Omega, \mathcal{F}, P)$ spaces. Jouini et al. (2006) demonstrate that a major assumption on continuity can be dropped in this case (the Fatou property), as it is automatically satisfied. Here we consider the analysis in $L_p(\Omega, \mathcal{F}, P)$ spaces, $p \in [1, \infty)$, although different Banach spaces of random variables can be considered equally well, cf. for example Bellini et al. (2014) in this journal. For an extended discussion of a natural choice of an appropriate Banach space we refer the reader to Pichler (2013b).

Outline of the paper. The following section provides the mathematical exposition. Section 3 introduces Kusuoka's representation and the measure preserving transportation map, which is a main tool of our analysis. The following section (Section 4) addresses premium principles on general probability spaces. Section 5 discusses maximality of Kusuoka sets, which can be used to characterize minimal representations by employing stochastic dominance relations of first and second order. Section 6 exposes examples of minimal representations of important premium principles, while we conclude in Section 7.

2. Mathematical setting

Throughout the paper we work with spaces $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$. That is, $Z \in \mathcal{Z}$ can be viewed as a random variable with finite p-th order moment with respect to the reference probability measure P. This is natural from an applications point of view and theoretically convenient. The space $\mathcal{Z}$ equipped with the norm $\|Z\|_p := \left(\int |Z|^p dP\right)^{1/p}$ is a Banach space with the dual space of continuous linear functionals $\mathcal{Z}^* := L_q(\Omega, \mathcal{F}, P)$, where $q \in (1, \infty]$ and $1/p + 1/q = 1$.

It is said that a (real valued) functional $\pi : \mathcal{Z} \to \mathbb{R}$ is a *coherent risk measure*[1] or a *coherent insurance premium*, if it satisfies the following axioms (Artzner et al., 1999):

(A1) *M*onotonicity: if $Z, Z' \in \mathcal{Z}$ and $Z \le Z'$, then $\pi(Z) \le \pi(Z')$.
(A2) *C*onvexity:

$$\pi(tZ + (1-t)Z') \le t\pi(Z) + (1-t)\pi(Z')$$

for all $Z, Z' \in \mathcal{Z}$ and all $t \in [0, 1]$.

(A3) *T*ranslation Equivariance: if $c \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\pi(Z + c) = \pi(Z) + c$.
(A4) *P*ositive Homogeneity: if $t \ge 0$ and $Z \in \mathcal{Z}$, then $\pi(tZ) = t\pi(Z)$.

It is said that the risk measure π is *convex* if it satisfies axioms (A1)–(A3). The notation $Z \le Z'$ expresses that the policy Z' pays more than Z, i.e., $Z(\omega) \le Z'(\omega)$ for a.e. $\omega \in \Omega$. For an introductory discussion of premium principles (coherent (convex) risk measures) we refer the reader to Young (2006).

We say that two policies $Z_1, Z_2 \in \mathcal{Z}$ are *distributionally equivalent* if $F_{Z_1} = F_{Z_2}$, where $F_Z(z) := P(Z \le z)$ denotes the cumulative distribution function (cdf) of $Z \in \mathcal{Z}$. The risk measure (premium) $\pi : \mathcal{Z} \to \mathbb{R}$ is *law invariant* (with respect to the reference probability measure P) if for any distributionally equivalent insurance policies $Z_1, Z_2 \in \mathcal{Z}$ it follows that $\pi(Z_1) = \pi(Z_2)$. An important example of a law invariant coherent risk measure is the *Conditional Tail Expectation* (also called Average Value-at-Risk, or Conditional Value-at-Risk),

$$\begin{aligned} \mathsf{CTE}_\alpha(Z) &:= \inf_{t\in\mathbb{R}}\left\{t + (1-\alpha)^{-1}\mathbb{E}[Z-t]_+\right\} \\ &= (1-\alpha)^{-1}\int_\alpha^1 F_Z^{-1}(\tau)d\tau, \end{aligned} \tag{2.1}$$

where $\alpha \in [0, 1)$ and $F_Z^{-1}(\tau) := \sup\{t : F_Z(t) \le \tau\}$ is the right side quantile function. Note that $F_Z^{-1}(\cdot)$ is a monotonically nondecreasing right side continuous function.

- We denote by $\mathfrak{P}$ the set of probability measures on $[0, 1]$ having zero mass at 1. Unless stated otherwise, when talking about topological properties of $\mathfrak{P}$ we use the *weak topology* of probability measures (see, e.g., Billingsley, 1968 for a discussion of weak convergence of probability measures).

It was elaborated by Kusuoka (2001) that when the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic, every law invariant, coherent risk measure π has the representation

$$\pi(Z) = \sup_{\mu\in\mathfrak{M}} \int_0^1 \mathsf{CTE}_\alpha(Z)d\mu(\alpha), \quad Z \in \mathcal{Z}, \tag{2.2}$$

where $\mathfrak{M}$ is a set of probability measures on $[0, 1]$.

Definition 2.1. We say that a set $\mathfrak{M}$ of probability measures on $[0, 1]$ is a *Kusuoka set* if the representation (2.2) holds.

Note that the Kusuoka set is associated with a risk measure π and the space $\mathcal{Z}$. Since it is assumed that $\pi(Z)$ is finite valued for every $Z \in L_p(\Omega, \mathcal{F}, P)$, with $p \in [1, \infty)$, it follows that every measure $\mu \in \mathfrak{M}$ in representation (2.2) has zero mass at $\alpha = 1$ and hence $\mathfrak{M} \subset \mathfrak{P}$. Note also that if $\mathfrak{M}$ is a Kusuoka set, then its topological closure is a Kusuoka set too (cf. Shapiro, 2013, Proposition 1).

Outline of the paper. The paper is organized as follows. In the next section we introduce the notation and discuss minimality and uniqueness of the Kusuoka representations. In Section 4 we consider Kusuoka representations on general, not necessarily nonatomic, probability spaces. In Section 5 we investigate maximality of Kusuoka representations with respect to order, or dominance relations. In Section 6 we discuss some examples, while Section 7 is devoted to conclusions.

3. Uniqueness of Kusuoka sets

It is known that the Kusuoka representation is not unique in general. In this section we elaborate that there is minimal Kusuoka representation in the sense outlined below. To obtain the result we shall relate the Kusuoka representation to distortion premiums (Wang premiums) first and then outline the results.

[1] In a financial context the term *coherent risk measure* is often used for mapping $\mathcal{R}(Z) = \pi(-Z)$, or the concave mapping $\mathcal{R}(Z) = -\pi(-Z)$ instead. The axioms then change accordingly.