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The uncertain premium principle based on the distortion function

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HIGHLIGHTS

- The net premium principle for uncertain risks is presented within the framework of uncertainty theory.
- The uncertain distortion premium principle is derived from the uncertain net premium and some properties are investigated.
- A characterization theorem for uncertain distortion premium principle is proved.
- Several numerical examples are given to illustrate the uncertain distortion premium principle.

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ABSTRACT

In this paper, we discuss the premium principle in uncertain environment. First, the net premium principle for uncertain risks is presented within the framework of uncertainty theory. With the help of distortion function, a new uncertain premium principle is derived from the uncertain net premium. Some properties of uncertain distortion premium principle are proved. Furthermore, a sufficient and necessary condition that an uncertain premium principle is an uncertain distortion premium principle has been characterized. Finally, some examples are given to illustrate the calculations of the uncertain distortion premium.

1. Introduction

In real life, there exist a lot of indeterminacy phenomena which usually bring about risks and lead to loss. In order to recompense the loss caused by risks, decision makers need to take measures when they face the indeterminacy phenomena. One of effective measures is to pay an insurer premium to cover the potential risks.

Traditional insurance takes the point of view that risk originates from randomness and regards risk as a non-negative random variable. Based on this view, probability theory is used to deal with the indeterminacy of randomness and risk. There has been a great momentum in research on insurance in stochastic environment. Borch (1974, 1990) applied the expected utility theory (Von Neumann and Morgenstern, 1944) to solve insurance questions. Bühlmann (1980, 1984) presented and extended a useful premium principle for risk in economics. Rolski et al. (1999) discussed insurance and finance with the help of stochastic processes. Wang (2000) proposed a class of distortion operations for pricing financial and insurance risks. Dhaene et al. (2002a,b) considered the theory and applications of comonotonicity in actuarial science and finance. Wu and Wang (2003) investigated a characterization of distortion premium principle. For more detailed information, please see the selected literature such as Denuit et al. (1999, 2005), Wu and Zhou (2006), Laeven and Goovaerts (2008), Lee et al. (2012) and Denuit and Dhaene (2012).

In insurance, in order to assign a suitable premium to an insurance risk, we usually need to obtain and analyze the loss data from this risk in the past. When the sample data is large enough, we may give a probability distribution that is close enough to the loss frequency in the long run. In this case, probability theory is a legitimate approach. However, we sometimes lack the observed data for some reasons. When we are lack of observed data, how do we deal with it? It seems that we have no choice but to invite some domain experts to evaluate the possible loss. Since human beings usually overweight unlikely events (Kahneman and Tversky, 1979), the estimated probability distribution based on experts' estimations may be far from the cumulative frequency. In this case, the law of large numbers is no longer valid and probability theory is no longer applicable. Recently Liu (2007, 2011) proposed an uncertain measure and developed an uncertainty theory which can be used to handle subjective indeterminacy quantity.

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Uncertainty theory is a consistent mathematical system which is suitable to cope with the indeterminacy, and especially suitable for the situation with subjective estimation or lack of historical data. Uncertainty theory provides a new approach to study the risk and insurance problems. Liu (2009) introduced the uncertainty theory in finance and promoted the research on uncertain financial markets. Liu (2010) discussed structural risk analysis and investment risk analysis in uncertainty environment. Liu (2013) built an uncertain insurance model and proved a ruin index theorem. Chen (2011) gave the America option pricing formulas within uncertainty theory. Peng and Yao (2011) presented a mean reversion uncertain stock model and corresponding option pricing formulas. Peng (2013) proposed two types of risk metrics of loss function for uncertain system.

In this paper, we describe the risk as a non-negative uncertain variable and mainly discuss the premium of uncertain risk within the framework of uncertainty theory. We will find that the uncertain net premium, as will be proposed, satisfies the linearity for independent uncertain risks. As a comparison, the net premium in stochastic insurance satisfies the additivity for all random losses (see Borch, 1974, 1990). Besides, the uncertain net premium principle is extended to a new uncertain premium principle based on the distortion function. Some properties of new uncertain distortion premium are discussed. We also provide a sufficient and necessary condition for an uncertain premium principle to be the uncertain distortion premium principle.

The rest of the paper is organized as follows. In Section 2, uncertainty theory is introduced in simple words and some basic properties are given. In Section 3, comonotone uncertain variables are defined and some useful properties are proved. Section 4 presents the uncertain net premium principle for uncertain risk. The uncertain premium based on distortion function is proposed in Section 5. Some properties of uncertain distortion premium are investigated in Section 6. A characterization theorem about uncertain distortion premium principle is provided in Section 7. Some examples are given in Section 8. Section 9 contains a brief summary.

2. Uncertainty theory

Let Γ be a nonempty set and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is assigned a number $\mathcal{M}\{\Lambda\}$. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu (2007) proposed the following three axioms:

(1) (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;
(2) (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
(3) (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Definition 2.1 (*Liu, 2007*)**.** The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms.

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertain space. In order to obtain an uncertain measure of compound event, Liu (2009) defined a product uncertain measure which produces the fourth axiom:

(4) (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertain space for $k = 1, 2, \ldots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure on the product σ-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots$ satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \inf_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}.$$

Definition 2.2 (*Liu, 2007*)**.** An uncertain variable is defined as a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 2.3 (*Liu, 2007*)**.** An uncertain variable ξ on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be (*a*) non-negative if $\mathcal{M}\{\xi < 0\} = 0$; and (*b*) positive if $\mathcal{M}\{\xi \le 0\} = 0$.

Definition 2.4 (*Liu, 2009*)**.** Let ξ and η be uncertain variables on $(\Gamma, \mathcal{L}, M)$. Then ξ and η are said to be independent if

$$\mathcal{M}\{\{\xi_1 \in B_1\} \cap \{\xi_2 \in B_2\}\} = \mathcal{M}\{\xi_1 \in B_1\} \wedge \mathcal{M}\{\xi_1 \in B_2\}$$

for any Borel sets B_1, B_2 of real numbers.

The uncertainty distribution $\Phi : \Re \to [0, 1]$ of an uncertain variable ξ is defined by Liu (2007)

$$\Phi(x) = \mathcal{M}\left\{\gamma \in \Gamma \middle| \xi(\gamma) \le x\right\}.$$

Peng and Iwamura (2010) has proved that a function is an uncertainty distribution function if and only if it is a non-decreasing function.

Definition 2.5 (*Liu, 2011*)**.** An uncertainty distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$.

Definition 2.6 (*Liu, 2011*)**.** Let ξ be an uncertain variable with regular distribution Φ. Then the inverse function Φ^{-1} is called the inverse uncertainty distribution function of ξ.

Definition 2.7 (*Liu, 2011*)**.** An uncertain variable ξ is called discrete if it takes values in $\{x_1, x_2, \ldots, x_n\}$ and

$$\Phi(x) = \begin{cases} \alpha_0, & \text{if } x < x_1 \\ \alpha_i, & \text{if } x_i \le x < x_{i+1},\ 1 \le i < n \\ \alpha_n, & \text{if } x \ge x_n, \end{cases}$$

denoted by $\mathcal{D}(x_1, \alpha_1, x_2, \alpha_2, \ldots, x_n, \alpha_n)$, where $x_1 < x_2 < \cdots < x_n$ and $0 = \alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n = 1$.

Definition 2.8 (*Liu, 2011*)**.** The uncertain variable

$$\xi_a = \begin{cases} 1 & \text{with uncertain measure } a \\ 0 & \text{with uncertain measure } 1 - a \end{cases}$$

is called Boolean uncertain variable, where a is a number between 0 and 1.

Definition 2.9 (*Liu, 2007*)**.** Let ξ be an uncertain variable. Then the expected value of uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \le r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

Let ξ be an uncertain variable with regular uncertainty distribution Φ. If the expected value $E[\xi]$ exists, Liu (2011) has proved

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$