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Characterizations of counter-monotonicity and upper comonotonicity by (tail) convex order

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HIGHLIGHTS

- Countermonotonic rvs characterized by minimal sum wrt convex order.
- A number of new properties of the tail convex order are developed.
- Upper comonotonic random vector shown to attain the maximal tail convex sum.
- Show the equivalence between maximal tail convex sum and additivity of risk measures.

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ABSTRACT

In this paper, we characterize counter-monotonic and upper comonotonic random vectors by the optimality of the sum of their components in the senses of the convex order and tail convex order respectively. In the first part, we extend the characterization of comonotonicity by Cheung (2010) and show that the sum of two random variables is minimal with respect to the convex order if and only if they are counter-monotonic. Three simple and illuminating proofs are provided. In the second part, we investigate upper comonotonicity by means of the tail convex order. By establishing some useful properties of this relatively new stochastic order, we prove that an upper comonotonic random vector must give rise to the maximal tail convex sum, thereby completing the gap in Nam et al. (2011)'s characterization. The relationship between the tail convex order and risk measures along with conditions under which the additivity of risk measures is sufficient for upper comonotonicity is also explored.
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1. Introduction

In the actuarial literature, it is a prominent result that within a Fréchet space of given marginal distributions, a comonotonic random vector gives rise to the largest sum with respect to the convex order (see, for example, Dhaene et al., 2002). Recently, the converse of this statement, namely the sufficiency of the maximal convex sum property for comonotonicity, was established by Cheung (2008) under a continuity assumption, which was dispensed with subsequently in Cheung (2010) by a convolution technique. Mao and Hu (2011) proved the same result by reducing the problem to a bivariate setting and utilizing the structure of the stop-loss transform. It is of interest to explore whether the properties of the sum of a random vector's components can also be used to characterize similar dependence structures, such as counter-monotonicity and upper comonotonicity.

Counter-monotonicity, the first of these two structures, has surprisingly received scant attention in the literature. Though it is the exact opposite of comonotonicity, studies on the fundamental properties of counter-monotonic random vectors are minimal. Interest in this concept, however, was rekindled in Cheung et al. (submitted for publication) in the context of merging risks. From the development of the theory of comonotonicity, it is reasonable to postulate that the minimality of the sum of the components of a bivariate random vector with respect to the convex order is equivalent to its counter-monotonicity.

For upper comonotonic random vectors, a characterization by the maximality of the sum of their components with respect to the tail convex order was put forward in Nam et al. (2011). However, their proof was not presented in full generality, in that the necessity part, namely, the fact that an upper comonotonic random vector achieves the maximal tail convex sum, was taken from a result in Cheung and Vanduffel (2013), in which some restrictive assumptions on the marginal distributions were made.

In the same spirit as Cheung (2010), this paper aims to enrich the theoretical development of the theory of counter-monotonicity

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and upper comonotonicity by giving characterizations of these two dependence structures via the convex order and tail convex order respectively. In Section 2, we summarize the results needed in the remainder of the paper regarding inverse distribution functions, stochastic orders and dependence structures. In Section 3, we extend Cheung (2010)'s characterization of comonotonic random vectors by the sum of their components to the case of counter-monotonic random vectors. After giving three short and equally instructive proofs of this key result, we discuss how the techniques illustrated in these proofs can be applied to simplify existing proofs of characterizations of comonotonicity. In Section 4, we study upper comonotonicity by means of the tail convex order, some new properties of which are proved: (1) the tail convex order is stable under upper comonotonic addition; (2) an order with respect to the tail convex order induces a corresponding order on the Tail Value-at-Risk for all sufficiently large levels of probability, vice versa. Making use of these new properties, we readily accomplish the necessity part of Nam et al. (2011)'s characterization in full generality and describe the maximal tail convex sum property in terms of the additivity of risk measures. Finally, through analyzing a counter-example in Nam et al. (2011), we highlight the alarming discovery that, in contrast to comonotonicity, in general the additivity of the Value-at-Risk and Tail Value-at-Risk is not a sufficient condition for upper comonotonicity. This can be attributed to the delicate geometry inherent in the support of an upper comonotonic random vector. Under the assumption of continuous marginal distributions, the equivalence between upper comonotonicity and the maximal tail convex sum property holds.

2. Preliminaries

Throughout this paper, we assume that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All distributions are assumed to be integrable.

2.1. Inverse distribution functions

For a given random vector $\mathbf{X} = (X_1, \dots, X_n)$, we denote its distribution function and survival function by $F_{\mathbf{X}}$ and $\overline{F}_{\mathbf{X}}$ respectively. If X is a random variable, then its inverse distribution function can be defined in different ways:

- Its left-continuous and right-continuous inverse distribution functions, F_X^{-1} and F_X^{-1+}, are defined respectively by

  $$F_X^{-1}(p) := \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in [0, 1],$$

  and

  $$F_X^{-1+}(p) := \inf\{x \in \mathbb{R} \mid F_X(x) > p\}, \quad p \in [0, 1].$$

- For any $\alpha \in [0, 1]$, the α-mixed inverse distribution function $F_X^{-1(\alpha)}$ is defined as

  $$F_X^{-1(\alpha)}(p) := \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p), \quad p \in [0, 1].$$

  Observe that $F_X\left(F_X^{-1(\alpha)}(p)\right) \geq p$ for any $\alpha \in [0, 1]$ and the inequality can be strict if F_X has a jump at $F_X^{-1(\alpha)}(p)$.

Denote by $\mathcal{R}(F_1, \dots, F_n)$ the Fréchet space of all random vectors with $F_1, \dots, F_n$ as marginal distributions.

2.2. Stochastic orders

Several notions of variability orders will be employed in the sequel. Standard references on the theory of stochastic orders include Muller and Stoyan (2002), Denuit et al. (2005) and Shaked and Shanthikumar (2007).

Definition 1. Let X and Y be two given random variables.

1. X is said to be smaller than Y in the convex order, written as $X \leq_{cx} Y$, if $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $d \in \mathbb{R}$. The relationship $X \leq_{cx} Y$ can be interpreted as the fact that X is less variable than Y. Obviously, $X \leq_{cx} Y$ implies $\mathrm{Var}(X) \leq \mathrm{Var}(Y)$, but not the converse. If the latter holds, then X is said to be less than Y in the variance order, denoted as $X \leq_{\mathrm{var}} Y$.
2. X is said to precede Y in the tail convex order, written as $X \leq_{tcx} Y$, if $\mathbb{E}[X] = \mathbb{E}[Y]$ and there exists a real number k, called the tail index, such that
   (i) $\mathbb{P}(Y > k) > 0$;
   (ii) $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $d \geq k$.

Tail convex order was first proposed in Cheung and Vanduffel (2013) in conjunction with the study of upper comonotonicity and further applied in Nam et al. (2011). An interpretation of $X \leq_{tcx} Y$ is that X is less variable than Y beyond the threshold k. Note that unlike Nam et al. (2011), in Condition (i) we merely require that there is a strictly positive probability that Y, rather than X, is greater than k. Such a definition of the tail convex order is more natural and meaningful for two reasons:

1. It allows the possibility that X is almost surely less than or equal to k, in which case Y, having a strictly positive probability to attain the level k or above, is still more variable than Y beyond k.
2. The imposition of Condition (i) avoids the triviality that any two bounded random variables X and Y can always be ordered. In this case, $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ holds whenever d is greater than the maximum of the essential suprema of X and Y, beyond which the comparison of the variability of X and Y is meaningless. Pathological instances like $U_1 \leq_{tcx} U_2$ with tail index 1, where U_1 is a uniform$(-1, 1)$ random variable and U_2 is a uniform$(-0.5, 0.5)$ random variable, are ruled out.

Under this refined definition, the tail convex order is transitive.

Lemma 1. *If $X \leq_{tcx} Y$ with tail index k_1 and $Y \leq_{tcx} Z$ with tail index k_2, then $X \leq_{tcx} Z$ with tail index $k_1 \vee k_2$, where $k_1 \vee k_2 := \max(k_1, k_2)$.*

Proof. It is clear that $\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[Z]$ and $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+] \leq \mathbb{E}[(Z-d)_+]$ for all $d \geq k_1 \vee k_2$. It remains to check Condition (i) in the definition of the tail convex order:

- If $k_1 \leq k_2$, then $\mathbb{P}(Z > k_1 \vee k_2) = \mathbb{P}(Z > k_2) > 0$ by definition.
- If $k_1 > k_2$ and $\mathbb{P}(Z > k_1) = 0$, then

  $$0 < \mathbb{E}[(Y-k_1)_+] \leq \mathbb{E}[(Z-k_1)_+] = 0,$$

  which is a contradiction.

In both cases, we have $\mathbb{P}(Z > k_1 \vee k_2) > 0$. □

2.3. Comonotonicity and its variants

Comonotonicity is an extreme form of dependence structure that describes the strongest positive dependence. A subset A in $\mathbb{R}^n$ is comonotonic if $(t_i - s_i)(t_j - s_j) \geq 0$ for any $(s_1, \dots, s_n)$ and $(t_1, \dots, t_n)$ in A and $i, j \in \{1, \dots, n\}$. If a random vector $(X_1, \dots, X_n)$ has a comonotonic support, it is called comonotonic.

Counter-monotonicity is the antithesis of comonotonicity. A pair of random variables (X, Y) is said to be counter-monotonic if $(X, -Y)$ is comonotonic. Note that this notion is well-defined only in two dimensions.

In the sequel, we denote the comonotonic and counter-monotonic counterparts of a given random vector (X_1, X_2) by (X_1^c, X_2^c) and (X_1^{cm}, X_2^{cm}) respectively.

For ease of exposition, we summarize all properties of comonotonicity and counter-monotonicity that will be useful to later developments of this paper in Lemma 2 below. They can be found, for example, in Dhaene et al. (2002) and Denuit et al. (2005).

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