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Moments of discounted aggregate claim costs until ruin in a Sparre Andersen risk model with general interclaim times

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HIGHLIGHTS

- We consider a Sparre Andersen risk model with arbitrary interclaim time distribution.
- The moments of discounted aggregate claim costs until ruin are studied.
- A novel generalization of the discounted density is proposed to analyze the problem.
- Explicit formulae are derived upon assumptions on claims in two detailed examples.
- Numerical illustrations are also given.

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ABSTRACT

In the context of a Sparre Andersen risk model with arbitrary interclaim time distribution, the moments of discounted aggregate claim costs until ruin are studied. Our analysis relies on a novel generalization of the so-called discounted density which further involves a moment-based component. More specifically, while the usual discounted density contains a discount factor with respect to the time of ruin, we propose to incorporate powers of the sum until ruin of the discounted (and possibly transformed) claims into the density. Probabilistic arguments are applied to derive defective renewal equations satisfied by the moments of discounted aggregate claim costs until ruin. Detailed examples concerning the discounted aggregate claims and the number of claims until ruin are studied upon assumption on the claim severities. Numerical illustrations are also given at the end.

1. Introduction

In this paper, we consider an insurance company whose surplus is modelled as a *Sparre Andersen (renewal) risk process* (Sparre Andersen, 1957). Mathematically, the surplus level of the company at time t is given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where $u = U(0) \geq 0$ is the initial capital of the insurer and $c > 0$ is the incoming premium rate per unit time. In the above definition, $\{N(t)\}_{t\geq 0}$ is the claim number process which is a renewal process defined through the sequence of independent and identically distributed (i.i.d.) positive interclaim times $\{V_i\}_{i=1}^{\infty}$ having common distribution as V. More specifically, V_1 represents the time of the first claim and V_i for $i = 2, 3, \ldots$ is the time between the $(i-1)$-th claim and the i-th claim. Therefore, if $T_n = \sum_{i=1}^{n} V_i$ denotes the time of the n-th claim for $n = 1, 2, \ldots$ with the convention that $T_0 = 0$, then $N(t) = \sup\{n \in \mathbb{N} : T_n \leq t\}$. Here we assume that V is continuous with density $k(\cdot)$. In addition, the claim sizes $\{Y_i\}_{i=1}^{\infty}$ are positive continuous random variables which form an i.i.d. sequence distributed as Y and independent of $\{V_i\}_{i=1}^{\infty}$, and the density of Y is denoted by $p(\cdot)$. The time of ruin of $\{U(t)\}_{t\geq 0}$ is defined to be $\tau = \inf\{t \geq 0 : U(t) < 0\}$ with $\tau = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. The positive security loading condition $cE[V] > E[Y]$ ensures not only that the process $\{U(t)\}_{t\geq 0}$ drifts to infinity in the long run but also that the ruin probability is less than 1 (e.g. Prabhu, 1998, Part I, Theorems 2 and 7).

The Gerber–Shiu function (or its special cases) proposed by Gerber and Shiu (1998) has been studied extensively in the literature in various Sparre Andersen risk models. Most studies were conducted under specific distributional assumption on the interclaim times. See e.g. Dickson and Hipp (2001), Li and Garrido (2004, 2005) and Gerber and Shiu (2005). For Sparre Andersen models

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with general interclaim times and phase-type claims, ruin probability results can be found in Asmussen and Albrecher (2010, Chapter IX.4), whereas Drekic et al. (2004) showed that the deficit at ruin is also phase-type. If instead the claims follow exponential or Coxian distribution, the Gerber–Shiu function was obtained by Willmot (2007) and Landriault and Willmot (2008) respectively. In the context of a general (dependent) Sparre Andersen risk model, Albrecher and Teugels (2006) obtained asymptotic results for the ruin probability when claims are light-tailed, while further generalizations of Gerber–Shiu function were studied by Cheung et al. (2010), Woo (2010), Cheung (2011) and Willmot and Woo (2012).

In this paper, we are interested in the Gerber–Shiu type function

$$\phi_{n_1,n_2,\delta}(u) = E\Bigg[e^{-n_1\delta\tau}\left(\sum_{k=1}^{N(\tau)} e^{-\delta T_k} f(Y_k)\right)^{n_2} \times w(|U(\tau)|)\, I\{\tau < \infty\}\Bigg| U(0)=u\Bigg], \quad u \geq 0, \tag{1.1}$$

where $w(\cdot)$ is the so-called penalty function which only depends on the deficit at ruin $|U(\tau)|$; $I\{A\}$ is the indicator function of the event A; and $\delta \geq 0$ can either be viewed as a force of interest or a Laplace transform argument. The variable $\sum_{k=1}^{N(\tau)} e^{-\delta T_k} f(Y_k)$ represents the total discounted claim costs until ruin, with $f(\cdot)$ being the 'cost' as a function of a given claim severity. Its expectation was studied by Cai et al. (2009) and Feng (2009a,b) who considered the compound Poisson risk process and the phase-type renewal model respectively. Our general analysis of $\phi_{n_1,n_2,\delta}(u)$ in Section 2 does not require any specific assumptions on the cost function $f(\cdot)$ or the penalty function $w(\cdot)$. However, when deriving explicit solutions in Section 3, assumption on $f(\cdot)$ (but not $w(\cdot)$) is typically needed. See item 1 of the procedure discussed near the end of Section 2.

The proposed function defined by (1.1) contains various interesting special cases as follows.

1. If $n_1 = 1$ and $n_2 = 0$, then $\phi_{1,0,\delta}(u) = E[e^{-\delta\tau} w(|U(\tau)|) \times I\{\tau < \infty\}|U(0) = u]$ becomes a Gerber–Shiu function in which the penalty only depends on the deficit.
2. If $w(\cdot) \equiv 1$, then $\phi_{n_1,n_2,\delta}(u) = E[e^{-n_1\delta\tau}(\sum_{k=1}^{N(\tau)} e^{-\delta T_k} f(Y_k))^{n_2} \times I\{\tau < \infty\}|U(0) = u]$ can be regarded as the *generalized moment* of the discounted claim costs. The ideas of generalized moments can be found in Badescu and Landriault (2008, Section 3.1) (see also Cheung, 2008, Eq. (D.10)) who considered the discounted dividends payable until ruin. By letting $n_1 = 0$, the 'ordinary' moment $\phi_{0,n_2,\delta}(u) = E[(\sum_{k=1}^{N(\tau)} e^{-\delta T_k} f(Y_k))^{n_2} \times I\{\tau < \infty\}|U(0) = u]$ can be retrieved, and this quantity was studied by Cheung and Feng (2013, Eq. (2.3)) under a different class of risk models with Markovian claim arrivals using theories in piecewise-deterministic Markov processes. However, the techniques therein do not apply in the present Sparre Andersen risk model with arbitrary interclaim times. If it is further assumed that $f(x) = x$, then $\phi_{0,n_2,\delta}(u) = E[(\sum_{k=1}^{N(\tau)} e^{-\delta T_k} Y_k)^{n_2} \times I\{\tau < \infty\}|U(0) = u]$ represents the n_2-th moment of discounted aggregate claims until ruin, and it is important to distinguish it from the moment of discounted aggregate claims until a fixed time considered by Léveillé and Garrido (2001a,b) also in a renewal risk process (see the end of Section 4).
3. If $\delta = 0$ and $w(\cdot) \equiv f(\cdot) \equiv 1$, then $\phi_{n_1,n_2,0}(u) = E[(N(\tau))^{n_2} \times I\{\tau < \infty\}|U(0) = u]$ (which is independent of n_1) represents the n_2-th moment of the number of claims until ruin (see e.g. Landriault et al., 2011, Dickson, 2012 and Frostig et al., 2012).

It is remarked that although the parameter n_1 mostly takes values 1 or 0 and has no physical interpretation in retrieving the above special cases, general integer values of n_1 are required in our analysis. See Remark 3 for more details.

The paper is organized as follows. In Section 2, we consider some general properties of the function $\phi_{n_1,n_2,\delta}(u)$ defined by (1.1). In particular, $\phi_{n_1,n_2,\delta}(u)$ is shown to satisfy a defective renewal equation. Our derivation is based on a novel generalization of the well-known discounted density which further involves the moment-based component $(\sum_{k=1}^{N(\tau)} e^{-\delta T_k} f(Y_k))^{n_2}$. These provide general guidance as to how explicit formulae for $\phi_{n_1,n_2,\delta}(u)$ can be obtained when assumptions on the distribution of the claim severities and the cost function $f(\cdot)$ are made. To illustrate how this works, Section 3 studies in detail (1) the moments of discounted aggregate claims until ruin for exponential claims, and (2) the moments of the number of claims until ruin when claims follow a combination of exponentials. Numerical examples will be the subject matter of Section 4, whereas Section 5 ends the paper with some concluding remarks.

2. General consideration

2.1. Classical integral equation by conditioning on the first event

Our first step of the analysis involves the classical approach of 'conditioning on the time V_1 and the amount Y_1 of the first claim'. We shall look at the variable inside the expectation of the quantity $\phi_{n_1,n_2,\delta}(u)$. Distinguishing whether ruin occurs upon the first claim leads us to

$$\begin{aligned}
&e^{-n_1\delta\tau}\left(\sum_{k=1}^{N(\tau)} e^{-\delta T_k} f(Y_k)\right)^{n_2} w(|U(\tau)|)\, I\{\tau < \infty\} \\
&= e^{-n_1\delta V_1}\left(e^{-\delta V_1} f(Y_1)\right)^{n_2} w(|U(V_1)|)\, I\{\tau = V_1\} \\
&+ e^{-n_1\delta[V_1+(\tau-V_1)]}\left(e^{-\delta V_1} f(Y_1) + e^{-\delta V_1}\sum_{k=2}^{N(\tau)} e^{-\delta(T_k-V_1)} f(Y_k)\right)^{n_2} \\
&\times w(|U(\tau)|)\, I\{V_1 < \tau < \infty\} \\
&= e^{-(n_1+n_2)\delta V_1} f^{n_2}(Y_1)\, w(Y_1 - [U(0) + cV_1]) \\
&\times I\{U(0) + cV_1 - Y_1 < 0\} + \sum_{j=0}^{n_2}\binom{n_2}{j} e^{-(n_1+n_2)\delta V_1} f^{n_2-j}(Y_1) \\
&\times I\{U(0) + cV_1 - Y_1 \geq 0\}\Bigg[e^{-n_1\delta(\tau-V_1)}\left(\sum_{k=2}^{N(\tau)} e^{-\delta(T_k-V_1)} f(Y_k)\right)^{j} \\
&\times w(|U(\tau)|)\, I\{\tau < \infty\}\Bigg],
\end{aligned} \tag{2.1}$$

which holds almost surely. Noting that the process restarts at level $U(0) + cV_1 - Y_1$ if the first claim does not cause ruin, we arrive at

$$\begin{aligned}
\phi_{n_1,n_2,\delta}(u) = &\int_0^\infty e^{-(n_1+n_2)\delta t}\left(\int_{u+ct}^\infty f^{n_2}(y) \times w(y - (u+ct))\, p(y)\, dy\right) k(t) dt \\
&+ \sum_{j=0}^{n_2}\binom{n_2}{j}\int_0^\infty e^{-(n_1+n_2)\delta t}\left(\int_0^{u+ct} f^{n_2-j}(y) \times \phi_{n_1,j,\delta}(u+ct-y)\, p(y)\, dy\right) k(t)\, dt.
\end{aligned} \tag{2.2}$$