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Corrected phase-type approximations of heavy-tailed risk models using perturbation analysis

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HIGHLIGHTS

- We construct approximations for the ruin probability under heavy-tailed claim sizes.
- We are the first to provide approximations with provably small absolute and relative errors.
- Our approximations have simple expressions, the correct tail behavior and error bounds.
- We ensure high accuracy for all values of the initial capital and the safety loading.

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ABSTRACT

Numerical evaluation of performance measures in heavy-tailed risk models is an important and challenging problem. In this paper, we construct very accurate approximations of such performance measures that provide small absolute and relative errors. Motivated by statistical analysis, we assume that the claim sizes are a mixture of a phase-type and a heavy-tailed distribution and with the aid of perturbation analysis we derive a series expansion for the performance measure under consideration. Our proposed approximations consist of the first two terms of this series expansion, where the first term is a phase-type approximation of our measure. We refer to our approximations collectively as *corrected phase-type approximations*. We show that the corrected phase-type approximations exhibit a nice behavior both in finite and infinite time horizon, and we check their accuracy through numerical experiments.

1. Introduction

The evaluation of performance measures of risk models is an important problem that has been widely studied in the literature (Asmussen and Albrecher, 2010; Klüppelberg et al., 2004; Kyprianou, 2006). Under the presence of heavy-tailed claim sizes, these evaluations become more challenging and sometimes even problematic (Ahn et al., 2012; Asmussen and Pihlsgård, 2005). In such cases, it is necessary to construct approximations for the quantity under consideration. In this paper, we develop a new method to construct reliable approximations for performance measures of heavy-tailed risk models. We use the classical risk model (perhaps outdated, but very well studied) as a context and vehicle to demonstrate our key ideas, which we expect to have a much wider applicability in insurance. We show that our approximations have a provably small absolute error, independent of the initial capital, and a small relative error. As additional test of performance we also consider the finite horizon aggregate loss model.

There are three main directions for approximating ruin probabilities: phase-type approximations, asymptotic approximations and error bounds. When the claim sizes follow some light-tailed distribution, a natural approach to provide approximations for the ruin probability with high accuracy is by approximating the claim size distribution with a phase-type one (Feldmann and Whitt, 1998; Sasaki et al., 2004; Starobinski and Sidi, 2000). We refer to these methods as *phase-type approximations*, because the approximate ruin probability has a phase-type representation (Asmussen, 1992; Ramaswami, 1990). However, in many financial applications, an appropriate way to model claim sizes is by using heavy-tailed distributions (Asmussen, 2003; Embrechts et al., 1997; Rolski et al., 1999). In these cases, the exponential decay of phase-type approximations gives a big relative error at the tail and the evaluation of the ruin probability becomes more complicated.

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When the claim size distribution belongs to the class of subexponential distributions (Teugels, 1975), which is a special case of heavy-tailed distributions, asymptotic approximations are available (Bahr, 1975; Borovkov and Foss, 1992; Embrechts and Veraverbeke, 1982; Olvera-Cravioto et al., 2011; Pakes, 1975). The main disadvantage of such approximations is that they provide a good fit only at the tail of the ruin probability, especially for small safety loading. Another stream of research focuses on corrected diffusion approximations for the ruin probability (Blanchet and Zwart, 2010; Silvestrov, 2004). A disadvantage of such asymptotic techniques is the requirement of finite higher moments for the claim size distribution.

Finally, results on error bounds (Kalashnikov, 2002; Vatamidou et al., 2012) indicate that such bounds are rather pessimistic, especially in terms of relative errors, and in the case of small safety loading. There exist also bounds with the correct tail behavior under subexponential claims (Kalashnikov and Norberg, 2002; Korshunov, 2011), but these bounds are only accurate at the tail. A conclusion that can be safely drawn from all the above is that, although the literature is abundant with approximations for the ruin probability in the case of light-tailed claim sizes, accurate approximations for the ruin probability in the case of heavy-tailed claim sizes are still an open topic.

Besides the ruin probability, a very popular tool in real-world applications to measure the operational risk is the Value at Risk (VaR) (Embrechts et al., 2004). For a given portfolio, a VaR with a probability level α and fixed time horizon is defined as the threshold value such that the loss on the portfolio over the given time horizon exceeds this value with probability $1 - \alpha$. It is of interest to quantify the operational risk through the statistical analysis of operational loss data (Embrechts and Samorodnitsky, 2002; Klugman et al., 2008) and provide error bounds for the aggregate loss probability (Cox et al., 2008). Similarly to the ruin probability, things become more complicated under the presence of heavy-tailed data (Embrechts et al., 1997).

In this paper, we develop approximations for ruin probabilities and total losses under heavy-tailed claims that combine desirable characteristics of all three main approximation directions. First, our approximations maintain the computational tractability of phase-type approximations. Additionally, they capture the correct tail behavior, which so far could only be captured by asymptotic approximations, and they have the advantage that finite higher-order moments are not required for the claim sizes. Last, they provide a provably small absolute error, independent of the initial capital, and a small relative error.

The idea of our approach stems from fitting procedures of the claim size distribution to data. Heavy-tailed statistical analysis suggests that for a sample with size n only a small fraction ($k_n/n \to 0$) of the upper-order statistics is relevant for estimating tail probabilities (Davis and Resnick, 1984; Hill, 1975; Resnick, 2007). More information about the optimal choice of the k_nth upper order statistic can be found in Haeusler and Teugels (1985). The remaining data set may be used to fit the bulk of the distribution. Since the class of phase-type distributions is dense in the class of all positive definite probability distributions (Asmussen, 2003), a natural choice is to fit a phase-type distribution to the remaining data set (Asmussen et al., 1996). As a result, a mixture model for the claim size distribution is a natural assumption. Thus, our key idea is to use a mixture model for the claim size distribution in order to construct approximations of the ruin probability that combine the best elements of phase-type and asymptotic approximations.

We now sketch how to derive our approximations when the claim size distribution is a mixture of a phase-type distribution and a heavy-tailed one. Interpreting the heavy-tailed term of the claim size distribution in the mixture model as perturbation of the phase-type one and using perturbation theory, we can find the ruin probability (total loss) as a complete series expansion. The first term of the expansion is the phase-type approximation of the ruin probability (total loss) that occurs when we "remove" the heavy-tailed claim sizes from the system, either by discarding them or by replacing them with phase-type ones. We consider the model that appears when all heavy-tailed claims are removed as the "base" model. Due to the two different approaches of removing the heavy-tailed claim sizes, the ruin probability (total loss) connects to two different base models and consequently to two different series expansions.

We show that adding the second term of the respective series expansions is sufficient to construct improved approximations, compared to their phase-type counterparts, the *discard* and the *replace approximations*, respectively. Since the second term of each series expansion works as a correction to its respective phase-type approximation, motivated by the terminology *corrected heavy traffic approximations* (Asmussen, 2003), we refer to our approximations as *corrected phase-type approximations*. Therefore, in this paper, we propose the *corrected discard approximation* and the *corrected replace approximation*. Both approximations have appealing properties: the corrected replace approximation tends to give better numerical estimates, while the corrected discard approximation is simpler and yields guaranteed upper and lower bounds. Last, we provide the form of the *corrected phase-type approximations* for the aggregate loss over a fixed time period, and we show that they have the same appealing properties also for finite time.

Within risk theory, some attention has been given to perturbed risk models; see Schmidli (1999) for a review and the recent paper of Huzak et al. (2004). However, the term "perturbation" in this area is used to denote the superposition of two risk processes. Contrary to other asymptotic techniques that use perturbation analysis to approximate the ruin probability (Blanchet and Zwart, 2010; Silvestrov, 2004), our approach is different; we apply perturbation to the claim sizes rather than the arrival rate.

The connection between ruin probabilities and the stationary waiting probability $\mathbb{P}(W_q > u)$ of a G/G/1 queue, where service times in the queueing model correspond to the random claim sizes, is well known (Asmussen, 2003; Asmussen and Albrecher, 2010). Thus, the corrected approximations can also be used to estimate the waiting time distribution of the above mentioned queue. Finally, since the reserve process of the classical risk model is a basic building block of any Lévy process (Klüppelberg et al., 2004; Kyprianou, 2006), and due to the connection of ruin probabilities with scale functions (Ahn et al., 2012; Biffis and Kyprianou, 2010), we expect that our technique is widely applicable to more general risk processes.

The rest of the paper is organized as follows. In Section 2, we introduce the model and we derive two series expansions for the ruin probability. From these series expansions we deduce approximations for the ruin probability, in Section 3, and we study their basic properties. In Section 4, we find the exact formula of the ruin probability for a specific mixture model and we study the extent of the achieved improvement when we compare our approximations with phase-type approximations of their related base model. In Section 5, we provide corrected phase-type approximations of the aggregate loss in finite time and we show through a numerical study that our approximations give excellent VaR estimates. Finally, in the Appendix, we give all the proofs.

2. Series expansions of the ruin probability

As proof of concept, we apply our technique to the classical Cramér–Lundberg risk model (Asmussen and Albrecher, 2010; Prabhu, 1961). In this model, we assume that premiums flow in