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Determination of the probability of ultimate ruin by maximum entropy applied to fractional moments

Henryk Gzyl [a,*], Pier-Luigi Novi-Inverardi [b], Aldo Tagliani [b]

[a] Centro de Finanzas, IESA; Caracas 1010, Venezuela

[b] Department of Computer and Management Sciences, University of Trento - 38100 Trento, Italy

HIGHLIGHTS

- We compare three methods for inverting the Laplace transform of the probability of ultimate ruin.
- A maximum entropy based method, a Fourier inversion based method and a probabilistic (moment based) method.
- The maximum entropy method uses a few fractional (eight in our case) moments as input.
- The probabilistic method needs very high number of moment for appropriate convergence.
- The Fourier inversion method used real values of the Laplace transform. All three agree quite well.

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ABSTRACT

In this work we present two different numerical methods to determine the probability of ultimate ruin as a function of the initial surplus. Both methods use moments obtained from the Pollaczek–Kinchine identity for the Laplace transform of the probability of ultimate ruin. One method uses fractional moments combined with the maximum entropy method and the other is a probabilistic approach that uses integer moments directly to approximate the density.

1. Introduction

The following (Cramer–Lundberg) model is a standard model in the insurance industry. Premiums are collected at a constant rate c and claims are paid according to a compound process $S(t) = \sum_{n\geq 1}^{N(t)} X_n$, where $N(t)$ is a Poisson process of intensity λ describing the total number of claims in $[0, t]$, the individual claims being described by a collection $\{X_n\}_{n\geq 0}$ of i.i.d. square integrable random variables, independent of $N(t)$. At time t the capital of the company is $C(t) = x + ct - S(t)$, and the time to ruin is defined as

$$T(x) = inf\{t \geq 0 \mid C(t) < 0\} \tag{1}$$

where x denotes the initial surplus. What actuarial scientists are after is the density $f_{T(x)}(t)$ of the c.d.f. $F_{T(x)}(t) \equiv \psi(x, t) = P(T(x) \leq t)$. Many books and many papers are devoted to the subject and give approaches to the estimation of this density. See for example Gerber and Shiu (1998) who determine an integro-differential equation for the joint density of the time to ruin, the surplus prior to ruin and the deficit at ruin. See also Rolski et al. (1999) where much of the modeling in insurance risk is described. A more modest objective consists in the study of $\psi(x) = \lim_{t\to\infty} \psi(x, t) \equiv P(T(x) < \infty)$ the probability of ultimate ruin.

A useful technique in applied mathematics consists of determining a probability density function from the knowledge of its Laplace transform. For the class of models that we consider, the Laplace transform of the ruin probability is provided by the Pollaczek–Kinchine identity, see for example the volume by Rolski et al. (1999). The identity states that

$$\hat{\psi}(\alpha) = \frac{1}{\alpha} - \frac{c - \lambda m}{c\alpha - \lambda(1 - \mathcal{L}_X(\alpha))}. \tag{2}$$

Here $\hat{\psi}(\alpha) = \int_0^\infty e^{-\alpha x}\psi(x)dx$, and $\mathcal{L}_X(\alpha) = E[e^{-\alpha X_1}]$. Also, $m = E[X_1]$.

Since the Laplace transform is not continuously invertible (i.e., the inverse exists, but it is not continuous), the inverse problem

* Corresponding author.
E-mail addresses: henryk.gzyl@iesa.edu.ve (H. Gzyl), pierluigi.noviinverardi@unitn.it (P.L. Novi-Inverardi), aldo.tagliani@unitn.it (A. Tagliani).

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consisting of numerically determining $\psi(x)$ is beset with difficulties. This is an important consideration if for example, $\mathcal{L}_X(\alpha)$ is to be estimated numerically. For then the errors may get amplified. Recently Albrecher et al. (2010) and Avram et al. (2011) discuss different methods for computing the inverse Laplace transform. The latter consider a numerical inversion procedure based on a quadrature rule that uses as stepping stone a rational approximation of the exponential function in the complex plane. The latter implement and review much of the work done using Padé approximants to invert the Laplace transform. The first of these approaches depends on special assumptions on the underlying distributions, and on the possibility of extending the Laplace transform to the complex plane. For the second, it may not be possible to guarantee the positivity of the obtained approximation, and the lack of convergence results does not provide the accuracy of the obtained results.

The method that we propose here uses only the values of the Laplace transform on the real axis, and therefore does not involve integration in the complex plane, and are convenient when the Laplace transform cannot be inverted analytically or cannot be approximated by a rational function.

As it is well known, Laplace transform inversion on the real line is a difficult, ill-conditioned, inverse problem. That conclusion is illustrated in literature through some spot examples. In the case where the underlying inverse Transform is a probability density function Gzyl et al. (2013) explicitly built a continuum of functions whose Laplace transforms differ each other by as little as we want, but such that inverse Transforms are completely different. Thus in the presence of measurement errors (or even in more extreme cases computer roundoff error) we can expect many, indeed infinitely possible solutions having different shape (continuous or with spikes). Then we cannot "filter out" spike behavior of the inverse Transform on the basis of numerical values of Laplace transform alone. The ambiguity may be removed if ground is provided for selecting one of the possible solutions (for instance, the ones which are essentially smooth) and rejecting the others. In other terms, we have to use knowledge of the structural behavior of inverse Transform to obtain numerical values. In our specific case the inverse transform coincides with the continuous positive function $\psi(x)$. Then MaxEnt density approximation used ((4) below) becomes meaningful and represents a suitable approximation of $\psi(x)$. MaxEnt method relies upon sound theoretical results of characterization and entropy convergence and guarantees a positive approximation of the underlying function $\psi(x)$. The main drawback associated to positivity preserving is represented by the computational cost. In the sequel we describe recent results to mitigate the computational cost.

The aim of this note is twofold: On one hand we want to promote the use of maximum entropy applied to fractional moments as data to reconstruct probability densities, and we apply it to a problem of interest in insurance. And on the other hand, we want to compare our proposal to a (theoretically) good method to reconstruct densities from the knowledge of their collection of integer moments. As both methods pick up values from the moment curve, are very different and use exact data, the comparison between methods has to rely upon several aspects, as input, computational effort and time, their output.

Preliminarily we observe that

Lemma 1.1. *With the notations introduced above*

$$\lim_{\alpha\to 0}\hat{\psi}(\alpha) = \hat{\psi}(0) = \int_0^\infty \psi(x)dx = \frac{\lambda E[X_1^2]}{2(c-\lambda m)}.$$

The proof can be seen in Asmussen and Albrecher (2010).

Consider now the auxiliary random variable A with density $f_A = \psi(x)/\hat{\psi}(0)$. Note that if we consider the random variable $Y = e^{-A}$, then the Laplace transform of f_A can be thought of as the moment curve of Y, that is

$$\hat{f}_A(\alpha) = E[e^{-\alpha A}] = E[Y^\alpha] \equiv \mu_Y(\alpha) = \frac{\hat{\psi}(\alpha)}{\hat{\psi}(0)}.$$

Certainly, once the probability density $f_Y(y)$ is determined, $f_A(x)$ is obtained by a simple change of variables. Thus the question becomes: Can we use moment based techniques to determine numerically $f_Y(y)$ from the knowledge of a finite collection of moments $\{\mu_Y(\alpha_i) \mid i = 1, \ldots, M\}$? We shall consider two lines of approach to solve this problem and compare the results with a standard Laplace inversion method.

The possibility of combining the maximum entropy method with fractional moments and how it improves on the use of integer moments, has been explored in Novi-Inverardi and Tagliani (2003) for more details on the method of choice of the fractional moments. We present a summary of it in Section 2.1.2 below.

How to use the integer moments directly, combined with a twist on a probabilistic version of Bernstein approximation method, has been proposed in Mnatsakanov (2011), who applied an idea proposed by Mnatsakanov and Ruymgaart (2003) to obtain the density of a probability distribution from the knowledge of its Laplace transform. We should also mention Mnatsakanov (2008a,b) in which the author explores issues related to the speed of convergence on the approximations involved.

The performance of the two methods is rated against a Fourier based method of inversion of the Laplace transform, which in this case is possible due to the fact that we have enough data (the knowledge of the Laplace transform of the unknown function). The conclusion of our study is that the maxentropic method is rather useful because it uses as input a small number of data points, thus minimizing instabilities in the inversion procedure.

The rest of the paper is organized as follows. In Section 2 we review the method of maximum entropy as well as the issue of how many moments to use and how to choose them. In Section 3 we review Mnatsakanov and Ruymgaart's technique. In Section 4 we briefly describe Crump's procedure to approximate the exact Fourier inversion formula, and in Section 5 we present the results of the implementation of the different approaches in the context of three examples. The examples differ in the nature of the singularities of the Laplace transform: in one case the transform can be inverted by inspection, in the other case we are confronted with a cut in the complex plane and in the last case, with an essential singularity. Finally in Section 6 we present some concluding remarks.

2. Reconstruction from fractional moments with the method of maximum entropy

We shall first review the method of maximum entropy and then examine the issue of how many and which moments to consider. The first question to attend to is whether the fractional moments determine a probability distribution. The issue is settled by the following two theorems by Lin (1992). Both results assert that a probability density can be determined from its fractional moments, and rely on the fact that an analytic function is determined by its values on a countable set having an accumulation point in the domain of analyticity.

Theorem 2.1 (*Lin*)**.** *Let F_Y be the distribution function of a positive random variable Y. Let $\{\alpha_n\}_{n\ge 0}$ be a sequence of positive and distinct numbers in $(0, a)$ for some $a > 0$, satisfying $\lim_{n\to\infty}\alpha_n = \alpha_0 < a$. If $E[Y^a] < \infty$, the sequence of moments $E[Y^{\alpha_n}]$ characterizes F_Y.*

Theorem 2.2 (*Lin*)**.** *Let F_Y be the distribution function of a random variable Y taking values in $[0, 1]$. Let $\{\alpha_n\}_{n\ge 0}$ be a sequence of positive and distinct numbers satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n\ge 1}\alpha_n = \infty$. Then the sequence of moments $E[Y^{\alpha_n}]$ characterizes F_Y.*