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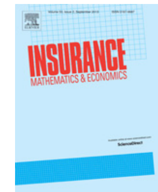

Finite time ruin probabilities for tempered stable insurance risk processes[☆]

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HIGHLIGHTS

- We study the family of tempered stable insurance risk processes.
- We derive a numerical approximation of a recent asymptotic representation of the ruin time distribution.
- Empirically the estimate provides a useful lower bound on the ruin distribution.
- Accurate estimates of the ruin time distribution can be obtained even for small initial capital.
- We derive a useful relationship between the parameters for safety loading management.

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ABSTRACT

We study the probability of ruin before time t for the family of tempered stable Lévy insurance risk processes, which includes the spectrally positive inverse Gaussian processes. Numerical approximations of the ruin time distribution are derived via the Laplace transform of the asymptotic ruin time distribution, for which we have an explicit expression. These are benchmarked against simulations based on importance sampling using stable processes. Theoretical consequences of the asymptotic formulae indicate that some care is needed in the choice of parameters to avoid exponential growth (in time) of the ruin probabilities in these models. This, in particular, applies to the inverse Gaussian process when the safety loading is less than one.

1. Introduction

The risk reserve of an insurance company has traditionally been modelled as a compound Poisson process with drift. In recent years more general Lévy processes have been proposed, among them the inverse Gaussian family of processes. Such processes have been found to approximate reasonably well a wide range of aggregate claims distributions (Chaubey et al., 1998). While the probability of eventual ruin has received a lot of attention, arguably of equal importance in practice is the probability of ruin before some finite time horizon. Our paper aims to study the probability of ruin before time t for the inverse Gaussian family and a natural generalisation, the tempered stable processes.

The basis of our investigation is the recent asymptotic representation, as the initial reserve grows large, of the ruin time distribution for more general "medium–heavy" convolution equivalent Lévy processes (Griffin, 2013; Griffin and Maller, 2012). This representation, via the calculation of its Laplace transform, lends itself to a numerical approximation of the ruin time distribution, which is then benchmarked against the values obtained by simulation. Thus we are able to illustrate the use of a broad, relatively simple and computationally tractable family of processes with which to model the risk reserve process.

We find that the asymptotic representation performs well even when the initial capital is relatively small, contrary to a view that asymptotic formulae may only be useful when the initial capital becomes extremely large. Additionally, the asymptotic representation provides some interesting insight with regard to safety loading management. Depending on the specific safety loading utilised in the insurance risk model, we show that processes within the

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tempered stable family may exhibit undesirable exponential growth (in time) of the ruin probabilities, at least asymptotically. This indicates that some caution may need to be exercised in the choice of model and to aid with this task, we derive a useful relationship between the parameters so as to avoid an unrealistic scenario. This might have interesting implications for practitioners concerned with safety loading management.

Empirically we also observe that the asymptotic formula provides a useful lower bound for the ruin probability that can be combined with the infinite horizon ruin probability to provide a practical approximation of the true ruin probability.

1.1. Lévy insurance risk model

Let $X = \{X_t : t \geq 0\}$, $X_0 = 0$, be a Lévy process defined on $(\Omega, \mathscr{F}, P)$, with canonical triplet $(\gamma_X, \sigma_X^2, \Pi_X)$. The characteristic function of X then has the Lévy–Khintchine representation $Ee^{\mathrm{i}\theta X_t} = e^{t\Psi_X(\theta)}$, where

$$\Psi_X(\theta) = \mathrm{i}\theta\gamma_X - \tfrac{1}{2}\sigma_X^2\theta^2 + \int_{\mathbb{R}} (e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|<1\}})\Pi_X(\mathrm{d}x),$$
$$\text{for } \theta \in \mathbb{R}. \tag{1}$$

In the *general Lévy insurance risk model*, the claim surplus process, which represents the excess in claims over income, is modelled by a Lévy process X with $X_t \to -\infty$ almost surely. Claims are represented by positive jumps, while premia and other income produce a downward drift in X. The insurance company starts with a positive reserve u, and ruin occurs if this level is exceeded by X. The assumption $X_t \to -\infty$ a.s. is a reflection of the premium being set to avoid certain ruin. This setup generalises the classical Cramér–Lundberg model in which

$$X_t = \sum_{i=1}^{N_t} U_i - pt, \tag{2}$$

where the nonnegative random variables U_i form an i.i.d. sequence with finite mean μ, N_t is an independent rate λ Poisson process, and $p > \lambda\mu$. Here U_i models the size of the ith claim and p represents the rate of premium inflow. The assumption $p > \lambda\mu$ is the *net profit condition* needed to ensure that $X_t \to -\infty$ a.s. See Asmussen and Albrecher (2010) and Embrechts et al. (1997) for background.

1.2. The convolution equivalent model

A natural class which includes the tempered stable distributions and the inverse Gaussian distribution is the class of *convolution equivalent distributions* of index α, which we now briefly describe. We will restrict ourselves to the non-lattice case, since this will be the main focus of this paper. The alternative can be handled by obvious modifications. A distribution F on $[0, \infty)$ with tail $\overline{F} = 1 - F$ belongs to the *class* $\mathscr{S}^{(\alpha)}$, $\alpha > 0$, if $\overline{F}(u) > 0$ for all $u > 0$,

$$\lim_{u\to\infty} \frac{\overline{F}(u+x)}{\overline{F}(u)} = e^{-\alpha x}, \quad \text{for } x \in (-\infty, \infty), \tag{3}$$

and

$$\lim_{u\to\infty} \frac{\overline{F^{2*}}(u)}{\overline{F}(u)} \text{ exists and is finite}, \tag{4}$$

where $F^{2*} = F * F$. Distributions in $\mathscr{S}^{(\alpha)}$ are called *convolution equivalent* with index α.

Basic results for convolution equivalent distributions and the corresponding convolution equivalent Lévy insurance risk processes are set out in detail in Klüppelberg et al. (2004) and Griffin and Maller (2012), and associated papers, so we only outline the main ideas here. A comparison of the medium–heavy convolution equivalent condition, the light-tailed Cramér condition ($Ee^{\nu_0 X_1} = 1$ for some $\nu_0 > 0$) and the heavy tailed subexponential condition can also be found in Griffin and Maller (2012).

A Lévy process is said to be *convolution equivalent*,[1] if

$$X_1^+ \in \mathscr{S}^{(\alpha)} \quad \text{for some } \alpha > 0. \tag{5}$$

The *convolution equivalent Lévy insurance risk model* is one in which

$$X_1^+ \in \mathscr{S}^{(\alpha)} \quad \text{for some } \alpha > 0 \quad \text{and} \quad X_t \to -\infty \ a.s. \tag{6}$$

Membership of $\mathscr{S}^{(\alpha)}$ is a property of the positive tail of the distribution of X_1. Condition (5) can equivalently be expressed in terms of the positive tail $\overline{\Pi}_X^+(u) = \Pi_X((u, \infty))$ of the Lévy measure (see Watanabe (2008)). Assuming $\overline{\Pi}_X^+(x_0) > 0$ for some $x_0 > 0$, so that X has positive jumps with probability 1, we say that $\overline{\Pi}_X^+ \in \mathscr{S}^{(\alpha)}$ if the same is true of the corresponding renormalised tail $(\overline{\Pi}_X^+(\cdot)/\overline{\Pi}_X^+(x_0)) \wedge 1$. With this understanding, (5) is equivalent to

$$\overline{\Pi}_X^+ \in \mathscr{S}^{(\alpha)} \quad \text{for some } \alpha > 0. \tag{7}$$

Convolution equivalent distributions of index α have exponential moments of order α, but of no larger orders. Thus, if ψ_X denotes the cumulant of X, so that
$Ee^{\beta X_t} = e^{t\psi_X(\beta)}$,
then $\psi_X(\beta)$ is finite if and only if $\beta \leq \alpha$.

Some asymptotic aspects of the model (2) where U_1 has a convolution equivalent distribution were recently considered by Tang and Wei (2010). In particular, explicit asymptotic formulae for the Gerber–Shiu function in the infinite horizon case were derived. Theoretical and numerical comparisons between models under the Cramér condition or a convolution equivalent condition were recently carried out in Griffin et al. (2012) for general Lévy insurance risk processes. It was observed that the "medium–heavy" regime transitions continuously into the "light-tailed" Cramér regime as certain parameters describing the models are varied. The convolution equivalent model was suggested as providing a broad and flexible apparatus for modelling the insurance risk process.

1.3. Eventual ruin

Convolution equivalent Lévy processes were introduced into risk theory in Klüppelberg et al. (2004). In addition to (7), Klüppelberg et al. (2004) assumed

$$Ee^{\alpha X_1} < 1. \tag{8}$$

Condition (8) implies that $(e^{\alpha X_t})_{t\geq 0}$ is a nonnegative supermartingale from which it follows that $X_t \to -\infty$ a.s., so the second condition in (6) is automatic in this case.

For a given initial reserve $u > 0$, the *ruin time* is defined by

$$\tau(u) = \inf\{t \geq 0 : X_t > u\}. \tag{9}$$

The main results in Klüppelberg et al. (2004) include the following asymptotic estimate for the probability of eventual ruin. Assume (7) and (8). Then

$$\lim_{u\to\infty} \frac{P(\tau(u) < \infty)}{\overline{\Pi}_X^+(u)} = \frac{Ee^{\alpha\overline{X}_\infty}}{-\psi_X(\alpha)}, \tag{10}$$

where

$$\overline{X}_t = \sup_{0\leq s\leq t} X_s. \tag{11}$$

This expression for the limit differs in form from that given in Klüppelberg et al. (2004), but is equivalent; see Remark 1. Under (8), $\psi_X(\alpha) < 0$ and $Ee^{\alpha\overline{X}_\infty} < \infty$. If $Ee^{\alpha X_1} \in [1, \infty)$ then $Ee^{\alpha\overline{X}_\infty} = \infty$, but $Ee^{\alpha\overline{X}_t} < \infty$ for all $t \geq 0$; see Lemma 2.1 in Griffin (2013).

[1] See Borovkov and Borovkov (2008) and Foss et al. (2011) for further background on subexponential and convolution equivalent distributions.