Insurance: Mathematics and Economics 55 (2014) 1–9

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Asymptotic analysis of risk quantities conditional on ruin for multidimensional heavy-tailed random walks

Jingchen Liu [a], Jae-Kyung Woo [b,*]

[a] Department of Statistics, Columbia University, 1255 Amsterdam Avenue, New York, NY, 10027, USA

[b] Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam Road, Hong Kong

HIGHLIGHTS

- Multidimensional renewal risk model with regularly varying increments is assumed.
- Asymptotic behaviors of risk-related quantities are studied.
- Total variation approximation results with method of Lyapunov inequality are used.

ARTICLE INFO

Article history:
Received April 2013
Received in revised form
August 2013
Accepted 28 November 2013

Keywords:
Multivariate regularly variation
Heavy-tailed increments
Hitting rare set
Lyapunov inequality

ABSTRACT

In this paper we consider a multidimensional renewal risk model with regularly varying claims. This model may be used to describe the surplus of an insurance company possessing several lines of business where a large claim possibly puts multiple lines in a risky condition. Conditional on the occurrence of ruin, we develop asymptotic approximations for the average accumulated number of claims leading the process to a rare set, and the expected total amount of shortfalls to this set in finite and infinite horizons. Furthermore, for the continuous time case, asymptotic results regarding the total occupation time of the process in a rare set and time-integrated amount of shortfalls to a rare set are obtained.

1. Introduction

A multidimensional risk model is a useful tool to portray the overall solvency of insurance companies with multiple lines of business that may be affected by common shocks. A catastrophic event usually causes severe losses and results in several types of casualty insurance claims. For instance, flood causes claims of both automobiles and houses. In this case, a multidimensional risk model attempts to capture the financial position of companies that can be significantly affected by risk events of such a grand scale.

To describe the tail dependence phenomena of large claims among different lines of business, we assume that the claim size follows a multivariate regularly varying distribution. This class of distributions is a popular choice to characterize dependence of extremal events. For instance, due to the tail dependence, if the claim size of one business line is huge, then there is a substantial probability that the claims of other lines are of comparable sizes. This phenomenon is often observed when the claims in different directions are caused by some common catastrophic events.

Let us start with a d-dimensional continuous time risk reserve vector process $\{U_t\}_{t\geq 0}$ representing the available reserve level of multiple business lines of an insurance company at time t. The claim arrivals follow a renewal process $\{N_t\}_{t\geq 0}$ with the interclaim times being a sequence of independent and identically distributed (i.i.d.) random variables $\{V_i\}_{i=1}^{\infty}$ with a general not necessarily exponential distribution, the generic random variable of which is denoted by V. The claim sizes, $\{Y_i\}_{i=1}^{\infty}$, independent of the interclaim times are a sequence of i.i.d. $\mathbb{R}^d$-valued regularly varying random vectors with index α denoted as $\mathrm{RV}(\alpha, \mu)$, where μ is the limiting measure of the distribution. A detailed description of multivariate regularly varying distributions is provided in Section 2.1. The insurer's reserve process at time t with initial reserve $R_0 \in \mathbb{R}^d$ is given by

$$U_t = R_0 + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0, \tag{1}$$

where $c \in (0,\infty)^d$ is a deterministic vector representing the premium rates. We further define the associated claim surplus

* Corresponding author. Tel.: +852 28578324.
E-mail addresses: jcliu@stat.columbia.edu (J. Liu), jkwoo@hku.hk, lions0217@gmail.com (J.-K. Woo).

0167-6687/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.insmatheco.2013.11.010

process

$$W_t = \sum_{i=1}^{N_t} Y_i - ct, \quad t \geq 0. \tag{2}$$

In this paper, we are concerned with events associated with ruin that occurs only at claim arrival times. It is sufficient to consider the reserve process at claim instances. Let us now consider a discrete time d-dimensional risk reserve vector process $\{R_n\}_{n=1}^{\infty}$ and a claim surplus process at the arrival of the nth claim $\{S_n\}_{n=1}^{\infty}$ defined as

$$R_n = U_{V_1+\cdots+V_n}, \qquad S_n = W_{V_1+\cdots+V_n}, \quad n = 1, 2, \ldots. \tag{3}$$

Furthermore, the discrete processes adopt a random walk structure

$$S_n = \sum_{i=1}^{n} X_i, \qquad R_n = R_0 - S_n = R_0 - \sum_{i=1}^{n} X_i, \quad n = 1, 2, \ldots,$$

where $\{X_i = Y_i - V_i c : i = 1, 2, \ldots\}$ is a sequence of i.i.d. increments. Let X be the generic random vector equal in distribution to X_i. We further assume $E(V^r) < \infty$ for some $r > \alpha$. Then, the regular variation of Y, that is, $Y \in \mathrm{RV}(\alpha, \mu)$, implies that $X \in \mathrm{RV}(\alpha, \mu)$ (e.g. Resnick, 2006). Therefore, S_n is a discrete time d-dimensional random walk with multivariate regularly varying increments.

We are primarily interested in the conditional distribution of the claim surplus process S_n given the ruin event that S_n falls into a certain ruin set. This ruin set denoted by A can be written as the union of finitely many half spaces of $\mathbb{R}^d$. Also, define the first passage time of S_n to the ruin set

$$\tau_A = \inf\{n \geq 0 : S_n \in A\}. \tag{4}$$

Similarly, for the continuous time claim surplus process, we define stopping time as

$$\xi_A = \inf\{t \geq 0 : W_t \in A\}. \tag{5}$$

We will later choose the ruin set A such that ξ_A coincides with the claim arrivals, that is,

$$\xi_A = \sum_{i=1}^{\tau_A} V_i \tag{6}$$

implying that the visit of set A is caused by claims and thus $\{\tau_A < \infty\}$ and $\{\xi_A < \infty\}$ are equivalent.

Our interest focuses on the asymptotic regime for which A deviates from the origin in some opposite direction where the random walk drifts. This corresponds to the situation that the initial reserve of each line of business is large. More precisely, we multiply the set A by a rarity parameter b. In practice, the parameter b should be determined by the initial reserve vector R_0. Then, the ruin set is $bA = \{b \cdot x : x \in A\}$ and ruin occurs if $\tau_{bA} < \infty$. Our study focuses primarily on the condition measure $P(\cdot|\tau_{bA} < \infty)$ as $b \to \infty$. Regularity conditions will be imposed to ensure that bA is a rare set. Indeed, in Section 2.2 conditions on the set A are presented excluding the cases in which the probability of hitting a ruin set equals 1 (i.e. safety loading condition, see e.g. Asmussen and Albrecher (2010, p. 3) and also Hult and Lindskog (2006) for the high dimensional setting). These notions will be given precisely in Sections 2.1 and 2.2.

Under the one-dimensional setting, the first passage time of random walk is a classic problem in applied probability and it has been studied intensively in many areas such as queueing theory, branching processes, and dam/storage processes. However, there are far fewer works on exact results under the multidimensional setting in the context of risk theory. With a relatively simple risk model (i.e. compound Poisson risk model), for example, Gong et al. (2012) consider a common shock model for a multidimensional risk process. Also, Badescu et al. (2011) study a two-dimensional risk model where two lines of business are connected in terms of a quota share reinsurance treaty and additionally one of them has its own aggregate claim process, which is a generalization of Avram et al. (2008). Within the risk theory context, other studies concerning multidimensional problems are also given by Chan et al. (2003), Cai and Li (2005, 2007), Yuen et al. (2006), Li et al. (2007), Avram et al. (2008), Dang et al. (2009), Rabehasaina (2009) and Czarna and Palmowski (2011). Whereas most of these studies deal with the problems in a two-dimensional compound Poisson model, the current work makes a generalization to the model with higher dimensions and more general claim arrival processes for heavy-tailed claim size distributions.

Furthermore, as discussed in Chapter XIII.9 of Asmussen and Albrecher (2010), asymptotic results related to a multidimensional ruin problem are to be found in other research areas. For example, Collamore (1996, 2002) investigate the asymptotic properties of the hitting probability of a rare set in a multidimensional random walk with light-tailed increments that involves an extension to the ruin problem studied in Cramér (1954). A related work in a multidimensional Lévy process is given by Dembo et al. (1994). Also, Hult et al. (2005) provide the asymptotic behavior of multivariate regularly varying random walks. We refer to Blanchet and Liu (2008) for the efficient computation, and to Blanchet and Liu (in press) for the asymptotic description of the conditional measure. Also, Hult and Lindskog (2006) deal with the heavy-tailed insurance portfolios of several lines of business with possible benefits from diversification effects between businesses.

In this paper, we seek to derive asymptotic results about conditional expectations given $\tau_{bA} < \infty$ of two risk quantities for both finite and infinite intervals. The first one is the expected accumulated number of claims leading the random walk S_n to a rare set that could be different from the ruin set bA. The other quantity is the expectation of the total amount of shortfall of S_n to a rare set. Note that when the quantities of our interest are defined in the discrete time case, the number of claim arrivals asymptotically plays the same role as time in the continuous case as the events $\{N_t \geq n\}$ and $\{V_1+\cdots+V_n \leq t\}$ coincide. In view of this discussion, for the continuous time claim surplus process (2), we retrieve the asymptotic results for the expectations of the total occupation time of W_t staying in the rare set and the time-integrated amount of shortfalls to the rare set by utilizing the results in the discrete time case. The precise definitions of these quantities will be provided in the subsequent section. In the literature, similar quantities are considered in the one-dimensional risk model. For example Loisel (2005) and Biard et al. (2010) study the expected time-integrated negative part of the process under the conditional measure given ruin in continuous time one-dimensional random walks; see also Gerber et al. (2012) considering the risk reserve process after the surplus is negative until bankruptcy occurs. For discrete-time compound Poisson processes, moments of the time to recovery up to non-ruin level zero were exploited in dos Reis (2000).

The contribution of this paper is two-fold. First, we generalize the one-dimensional results to the multivariate regularly varying random walks. The current setup provides a natural mechanism for incorporating the extremal dependence among the claims under the heavy-tailed setting. Second, the risk quantities are of more general forms and thus require different techniques than those in the literature to analyze them. Concerning the rare set that is used to define the risk quantities, it is assumed to be possibly different from the ruin set bA. This setting provides more flexibility of implementing the risk quantities under different frameworks of regulatory jurisdiction or internal rules. For example, some changes on the definition of the rare set may be necessary corresponding to the imposition of strong regulatory constraints. This is discussed further in Section 2.1. Under such a setting, the usual exercise of Fubini's theorem is not applicable. Hence we apply a total variation approximation result for the conditional measure $P(\cdot|\tau_{bA} < \infty)$