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Capital requirements with defaultable securities[☆]

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HIGHLIGHTS

- We study capital requirements with general acceptance sets and general eligible assets.
- We show that general capital requirements need not be finitely valued or continuous.
- We establish characterizations of finiteness and continuity.
- We provide applications to capital requirements based on Value-at-Risk and Tail Value-at-Risk.
- We show the nonexistence of "optimal" eligible assets.

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ABSTRACT

We study capital requirements for bounded financial positions defined as the minimum amount of capital to invest in a chosen *eligible* asset targeting a pre-specified *acceptability* test. We allow for general acceptance sets and general eligible assets, including defaultable bonds. Since the payoff of these assets is not necessarily bounded away from zero, the resulting risk measures cannot be transformed into cash-additive risk measures by a change of numéraire. However, extending the range of eligible assets is important because, as exemplified by the recent financial crisis, assuming the existence of default-free bonds may be unrealistic. We focus on finiteness and continuity properties of these general risk measures. As an application, we discuss capital requirements based on Value-at-Risk and Tail-Value-at-Risk acceptability, the two most important acceptability criteria in practice. Finally, we prove that there is no optimal choice of the eligible asset. Our results and our examples show that a theory of capital requirements allowing for general eligible assets is richer than the standard theory of cash-additive risk measures.

1. Introduction

The objective of this paper is to investigate capital requirements for bounded financial positions in a world where default-free securities do not necessarily exist. As we will see, this general problem cannot be treated within the standard theory of cash-additive risk measures. Hence, our work extends and complements the literature on cash-additive risk measures on spaces of bounded measurable functions. By doing so we hope to also contribute to a more informed application of the theory of risk measures to capital adequacy issues arising in the design of modern solvency regimes.

Liability holders of a financial institution are credit sensitive. They, and regulators on their behalf, are concerned that the institution may fail to fully honor its future obligations. This will be the case if the institution's *financial position*, or *capital position* – the value of assets net of liabilities – becomes negative in some future state of the economy. To address this concern financial institutions

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hold *risk capital* whose function is to absorb unexpected losses thereby reducing the likelihood that they may become insolvent. A key question in this respect is how much capital a financial institution should be required to hold to be deemed adequately capitalized by the regulator.

This type of question is best framed using the concepts of an *acceptance set* and of a *risk measure*. Coherent acceptance sets and coherent risk measures were introduced in the seminal paper by Artzner et al. (1999) in the context of finite sample spaces and by Delbaen (2002) for general probability spaces. Convex risk measures were studied by Föllmer and Schied (2002) and by Frittelli and Rosazza Gianin (2002). Since then the theory of risk measures has established itself as the standard theoretical framework to approach the problem of capital adequacy in financial institutions, and continues to influence the debate on modern solvency regimes in both the insurance and the banking world.

In this paper we work within a one-period model with dates $t = 0$ and $t = T$, and assume financial positions at time T belong to the space $\mathscr{X}$ of bounded measurable functions $X : \Omega \to \mathbb{R}$ on a given measurable space $(\Omega, \mathcal{F})$. At the core of the theory is the concept of an acceptance set $\mathscr{A} \subset \mathscr{X}$, representing the set of future capital positions corresponding to financial institutions that are deemed to be well capitalized. Once an acceptability criterion has been defined it is natural to ask whether the management of a badly capitalized financial institution can achieve acceptability by implementing appropriate actions and, if so, at what cost. The theory of risk measures was designed to answer this question for a particular type of management action: raising capital and investing in a reference traded asset, the so-called *eligible* asset.

In this framework, required capital is defined as the minimum amount of capital that, when invested in the eligible asset, makes a given financial position acceptable. Formally, if $\mathscr{A} \subset \mathscr{X}$ is the chosen acceptance set and $S = (S_0, S_T)$ represents a traded asset with initial value $S_0 > 0$ and positive payoff $S_T \in \mathscr{X}$, the risk capital required for a position $X \in \mathscr{X}$ is given by

$$\rho_{\mathscr{A},S}(X) := \inf\left\{m \in \mathbb{R}; X + \frac{m}{S_0}S_T \in \mathscr{A}\right\}. \tag{1.1}$$

The bulk of the literature on capital requirements has focused on *cash-additive* risk measures

$$\rho_{\mathscr{A}}(X) := \rho_{\mathscr{A},B}(X) = \inf\{m \in \mathbb{R}; X + m1_\Omega \in \mathscr{A}\}, \tag{1.2}$$

for which the eligible asset $B = (B_0, B_T)$ is a risk-free bond with $B_0 = 1$ and $B_T = 1_\Omega$. This case is in fact more general than it may seem at first. Indeed, consider the situation where the payoff S_T of the eligible asset is bounded away from zero and choose S as the numéraire. With respect to this new numéraire, any financial position $X \in \mathscr{X}$ and the acceptance set $\mathscr{A}$ take their "discounted" form $\widetilde{X} := X/S_T$ and $\widetilde{\mathscr{A}} := \{X/S_T; X \in \mathscr{A}\}$, respectively, and

$$\rho_{\mathscr{A},S}(X) = S_0\, \rho_{\widetilde{\mathscr{A}}}(\widetilde{X}). \tag{1.3}$$

The risk measure $\rho_{\mathscr{A},S}$ can be therefore expressed in terms of the cash-additive risk measure $\rho_{\widetilde{\mathscr{A}}}$.

Our starting point is the following observation: *The artifice of changing the numéraire does not work for all choices of the eligible asset.* Indeed, assume the eligible asset $S = (S_0, S_T)$ is a defaultable bond with price $S_0 \in (0, 1)$ and face value 1. Since the bond is defaultable, in some future state $\omega \in \Omega$ the payoff $S_T(\omega)$ may be strictly smaller than the face value. Hence, S_T is a random variable taking values in the interval $[0, 1]$ and represents the "recovery rate", i.e. the portion of the face value that is recovered by the bond holder. If S_T is not bounded away from zero, which includes the possibility that it is zero in some future state, the risk measure $\rho_{\mathscr{A},S}$ *cannot be reduced to a cash-additive risk measure* by changing the numéraire. If the recovery rate is always strictly positive but not bounded away from zero, the discounted positions $\widetilde{X} := X/S_T$ make sense but no longer belong to $\mathscr{X}$ in general. Hence, we can no longer operate in the space of bounded measurable functions and the resulting underlying space will depend on the choice of the numéraire. If, on the other hand, the recovery rate is zero in some future scenario, then it is not even meaningful to speak about discounted positions. As a consequence, focusing on cash-additive risk measures only, rules out the possibility that the eligible asset is a general defaultable bond.

As the recent financial crisis has made painfully clear, assuming the existence of default-free bonds may turn out to be delusive. Indeed, even recovery rates which are zero in some future scenario are not unrealistic. Zero recovery rates arise naturally also in situations where actual recovery is strictly positive but may come too late to be of practical relevance in the capital adequacy assessment. In such cases, for solvency purposes, it may be necessary to assume zero recovery. Hence, to obtain a more realistic theory of capital requirements allowing for the possibility that the eligible asset is a defaultable bond, *we are forced to go beyond cash additivity and consider risk measures $\rho_{\mathscr{A},S}$ with respect to general eligible assets.*

Given an acceptance set $\mathscr{A}$ and a general eligible asset $S = (S_0, S_T)$ we will focus on the following three questions:

1. When is $\rho_{\mathscr{A},S}$ finite-valued? This is an important question, also from an economic perspective. Indeed, if $\rho_{\mathscr{A},S}(X) = -\infty$ for a position $X \in \mathscr{X}$, then we can extract arbitrary amounts of capital from the financial institution while retaining acceptability, which is clearly not economically meaningful. If, on the other hand, $\rho_{\mathscr{A},S}(X) = \infty$, then X cannot be made acceptable no matter how much capital we raise and invest in the eligible asset. Hence, the choice of the eligible asset is not "effective" when it comes to modifying the acceptability of X.
2. When is $\rho_{\mathscr{A},S}$ continuous? This is also of practical relevance since typically capital positions are based on estimates and can only be assessed in an approximate manner. Thus, it is important to know whether capital requirements are "stable" with respect to small perturbations of the capital position.
3. Is it possible to find an optimal eligible asset leading to the lowest risk capital compatible with a given acceptability criterion? This is related to the "efficiency" of the choice of the eligible asset, i.e. to the ability to reach acceptance with the least possible amount of capital.

In addressing the first two questions we will find that if we allow for general eligible assets, the range of possible behaviors is much broader than in the standard cash-additive setting, where every risk measure on $\mathscr{X}$ is finite-valued and globally Lipschitz-continuous. We will exhibit examples of capital requirements which are neither finite-valued nor continuous. This is even the case when the underlying acceptability test is based on Value-at-Risk or Tail Value-at-Risk, which are the two typical choices in modern regulatory environments.

As a consequence of general results on finiteness (Theorem 3.3) and continuity (Theorems 4.2 and 4.4), we will show that capital requirements based on Value-at-Risk are not always finite-valued (Proposition 3.6), and, even when finite-valued, are not always continuous (Proposition 4.13). Also capital requirements based on Tail Value-at-Risk need not be finite-valued (Proposition 3.9), but, whenever finite, they are also continuous (Proposition 4.17). In fact, for capital requirements based on Value-at-Risk and Tail-Value-at-Risk acceptability we will provide complete characterizations of finiteness and continuity. In the case that the eligible asset is a defaultable bond, these characterizations show that finiteness and continuity depend on the extent to which the issuer of the bond can default. In particular, our results show that, when the underlying probability space is nonatomic, capital