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Optimal reinsurance and investment with unobservable claim size and intensity

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HIGHLIGHTS

- Optimal reinsurance/investment problem in an unobservable risk model is studied.
- The intensity and jump size distribution are not informed.
- The closed form expressions of the optimal strategies are derived.
- The effect of the safety loading on the optimal strategies is investigated.

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ABSTRACT

We consider the optimal reinsurance and investment problem in an unobservable Markov-modulated compound Poisson risk model, where the intensity and jump size distribution are not known but have to be inferred from the observations of claim arrivals. Using a recently developed result from filtering theory, we reduce the partially observable control problem to an equivalent problem with complete observations. Then using stochastic control theory, we get the closed form expressions of the optimal strategies which maximize the expected exponential utility of terminal wealth. In particular, we investigate the effect of the safety loading and the unobservable factors on the optimal reinsurance strategies. With the help of a generalized Hamilton–Jacobi–Bellman equation where the derivative is replaced by Clarke's generalized gradient as in Bäuerle and Rieder (2007), we characterize the value function, which helps us verify that the strategies we constructed are optimal.

1. Introduction

In the past decade, optimal reinsurance and optimal investment problems for various risk models have gained a lot of interest in the actuarial literature; see e.g. Browne (1995), Hipp and Plum (2000), Schmidli (2001), Gaier et al. (2003), Hipp and Schmidli (2004), Yang and Zhang (2005), Liang (2007), Gu et al. (2010), Liang et al. (2011a), and references therein.

There are two trends in this literature. In the first one the insurance risk is modeled by continuous processes. For example, Promislow and Young (2005) modeled the risk as Brownian motion with drift and obtained an analytical expression for the minimum ruin probability and the corresponding optimal controls, when the insurance company also trades in a financial market; see also Luo et al. (2008). Bai and Guo (2008) took this one step further and considered the optimal reinsurance problem, with multiple risky assets and no-shorting constraint, obtained parallel results to Bayraktar and Young (2007) and showed that the optimal strategies for maximizing the expected exponential utility and minimizing the probability of ruin, are equivalent in some special cases. Gu et al. (2010) examined a variation of this problem when the risky asset follows a constant elasticity of variance model.

The second trend is using processes with jumps to model the insurance risk. For a compound Poisson risk model, Schmidli (2002) obtained the optimal strategy for the problem of minimizing the ruin probability. Under the criterion of maximizing the expected utility of terminal wealth, Irgens and Paulsen (2004) studied the optimal controls of reinsurance and investments for insurance portfolios with the return of risky asset being a jump–diffusion process. Liang et al. (2011b) discussed the optimal proportional reinsurance and investment problem to the case that the instantaneous rate of investment return follows an Ornstein–Uhlenbeck process. Besides, Liang and Guo (2008) found the optimal strategy

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which maximizes the adjustment coefficient for the compound Poisson risk processes. For further deliberations see Liu and Ma (2009) and Zhang and Siu (2012).

As suggested by Asmussen and Albrecher (2010, Chapter VII, p. 165), we model the environment by a Markov chain, i.e., take the intensity and the claim process to be modulated by a Markov chain. We further assume that the environment process is not directly observable and has to be inferred from the observation that the insurance company has from claims. Hidden Markov models have been used in optimal investment problems before but these were used to model the asset prices not the insurance risk as in this paper. A notable exception is Bayraktar and Poor (2008), in which the optimal time to change premiums is analyzed by formulating the problem as a quickest change detection problem. The rest of the literature mostly models the fact that the drift of the traded asset is not directly observable; see Lakner (1995, 1998), Sass and Haussmann (2004), Rieder and Bäuerle (2005), Bai and Guo (2007) and Liang et al. (2011b). Bäuerle and Rieder (2007) is an exception. This paper solves an optimal investment problem for a financial asset which follows a geometric Brownian motion with Poisson jumps with unobservable intensity. In the problem we consider, we are going to model the insurance risk process as a compound Poisson process (while the asset price process follows a geometric Brownian motion), with both unobserved intensity and claim size.

The main objective of our paper is to solve the optimal proportional reinsurance and investment problem for an unobservable Markov-modulated compound Poisson risk model, where the intensity and jump size distribution are not directly observable. This provides an important extension to the current set of models as the nature of claims (which we model as a hidden Markov chain) changes over time and the insurance company would have to react to those changes by modifying the reinsurance policy. The insurance company can explicitly filter out the Markov chain (i.e. derive the posterior probability distribution of the states of the Markov chain) using the observed claim arrivals using Lemma 3.1, and respond to the changes by implementing the optimal reinsurance policies described in Sections 4.2 and 4.3. By analyzing the effects of safety loading on the optimal strategies we are determining the maximum amount the reinsurance company charges for the risk it is taking. Using a recently developed results from filtering theory (see Bayraktar and Ludkovski, 2009), we reduce the partially observable control problem to an equivalent problem with complete observations. (For other recent methodological developments on the filtering problem of jump processes, also see Elliott et al., 2010, Elliott and Siu, 2012.) Our main result is on determining the optimal reinsurance problem and investigating the effect of the safety loading and the unobservable factors on the optimal reinsurance strategies. With the help of a generalized Hamilton–Jacobi–Bellman equation where the derivative is replaced by Clarke's generalized gradient as in Bäuerle and Rieder (2007), we verify the optimality of the optimal investment and reinsurance strategies we propose.

The rest of the paper is organized as follows. In Section 2, the model and assumptions are given. In Section 3, we reduce the partially observable control problem to an equivalent problem with complete observations, and derive the generalized Hamilton–Jacobi–Bellman equation. In Section 4, we investigate the existence and uniqueness of the optimal reinsurance strategies under both expected value principle and variance principle and obtain the optimal reinsurance policies. Here, we also analyze the effect of safety loading on the optimal reinsurance policies (in Sections 4.2 and 4.3). In Section 4.4, we investigate the influence of the unobservable factors on the optimal reinsurance strategies, and find that the optimal reinsurance strategy in the risk model with unknown jump intensity is always less or equal to the one in the risk model with known intensity. Using the notion of generalized Hamilton–Jacobi–Bellman equation we characterize the value function, and prove that the optimal investment and reinsurance policies we propose are optimal (in Section 4.5).

2. Model formulation

Under the Markov-modulated compound Poisson risk model, the surplus process for the insurance company is given by

$$dX_t = cdt - dS_t, \tag{2.1}$$

where c is the premium rate at time t and

$$S_t = \int_0^t \sum_{i\in E} 1_{\{M_s=i\}} dS_s^{(i)},$$

represents the aggregate claims up to time t. Here, $S^{(1)}, \ldots, S^{(d)}$ are independent compound Poisson processes with intensities and jump size distributions $(\lambda_1, \nu_1), \ldots, (\lambda_d, \nu_d)$, respectively. We also define the total measure $\mathfrak{v} = \nu_1 + \cdots + \nu_d$, and denote by $f_i(\cdot)$ the density of ν_i with respect to $\mathfrak{v}$. The process (M_t) in the expression for the aggregate claims is a continuous-time Markov chain with state space $E = \{e_1, \ldots, e_d\}$, where e_k is the k-th unit vector in R^d.

We denote by $Q_0 = (q_{ij})$ the infinitesimal generator of M. Therefore, $(\bar{\lambda}_t, \bar{\nu}_t) := ((\lambda, \nu)' M_t)$ where $(\bar{\lambda}, \bar{\nu}) = ((\lambda_1, \nu_1), \ldots, (\lambda_d, \nu_d)) \in R_+^d \times R_+^d$, i.e., as long as $M_t = e_j$, jumps arrive at rate λ_j and jump size distribution ν_j. Without loss of generality we will assume that

$$\lambda_1 \int y f_1(y)dy \le \lambda_2 \int y f_2(y)dy \le \cdots \le \lambda_d \int y f_d(y)dy. \tag{2.2}$$

Denote by $\sigma_0 = 0, \sigma_1, \sigma_2, \ldots$ the jump time points of the Poisson process S_t,

$$\sigma_l := \inf\{t > \sigma_{l-1} : S_t \neq S_{t-}\}, \quad l \ge 1$$

and by $Y_1, Y_2, \ldots$ the R-valued marks observed at these arrival times:

$$Y_l = S_{\sigma_l} - S_{\sigma_{l-}}, \quad l \ge 1.$$

As usual, we assume that the claim amounts Y_l is independent of claim-number process as well as M_t, and bounded from above with $Y_l > 0$. In terms of the counting random measure

$$N((0, t], A) := \sum_{l=1}^{\infty} 1_{\{\sigma_l \le t\}} 1_{\{Y_l \in A\}},$$

where A is a Borel set in R, we can write the observation process S_t as

$$S_t = \int_0^t \int y N(ds, dy).$$

In what follows we will assume that the insurer only knows the distribution of M_0, but is not informed about the intensity λ_i and jump size distributions ν_i, $i = 1, \ldots, d$.

Further, we allow the insurance company to continuously reinsure a fraction of its claim with the retention level $q_t \in [0, 1]$, and the reinsurance premium rate at time t is $\delta(q_t)$. Moreover, the company is allowed to invest its surplus in a financial market consisting of a risk-free asset (bond or bank account) and a risky asset (stock or mutual fund). Specifically, the price process of the risk-free asset is given by

$$dR_t = rR_t dt, \quad r > 0,$$

where r is the risk-free interest rate. A commonly-used model for stock price is that it follows a geometric Brownian motion. That is, the price P_t of a stock satisfies a stochastic differential equation

$$dP_t = aP_t dt + \sigma P_t dW_t,$$