Insurance: Mathematics and Economics 55 (2014) 191–199

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Properties of a risk measure derived from the expected area in red

Stéphane Loisel [a], Julien Trufin [b,*]

[a] Université de Lyon, Université Claude Bernard Lyon 1, Institut de Science Financière et d'Assurances, 50 Avenue Tony Garnier, F-69007 Lyon, France
[b] École d'Actuariat, Université Laval, Québec, Canada

ARTICLE INFO

Article history:
Received October 2013
Received in revised form January 2014
Accepted 27 January 2014

Keywords:
Ruin probability
Risk measure
Expected area in red
Stochastic ordering
Risk limit

ABSTRACT

This paper studies a new risk measure derived from the expected area in red introduced in Loisel (2005). Specifically, we derive various properties of a risk measure defined as the smallest initial capital needed to ensure that the expected time-integrated negative part of the risk process on a fixed time interval $[0, T]$ (T can be infinite) is less than a given predetermined risk limit. We also investigate the optimal risk limit allocation: given a risk limit set at a company level for the sum of the expected areas in red of all lines, we determine the way(s) to allocate this risk limit to the subsequent business lines in order to minimize the overall capital needs.

1. Introduction and motivation

Over the last decade, the concept of risk measures has become very popular in the insurance industry, especially with the introduction of the European Solvency II regulation. Insurance companies are now often required to hold a level of available capital such that the probability for economic ruin after one year is less than 1 in 200. The Value-at-Risk (VaR) has then emerged as a key instrument in insurance to compute the solvency capital requirement. See e.g. McNeil et al. (2005) for details on the VaR. Readers interested in articles dealing with regulation and solvency for insurance companies could see, e.g., Volume 35, Issue 1 of the Geneva Papers on Risk and Insurance — Issues and Practice. Although this new regulatory framework is an improvement compared to the old insurance practices, it does not consider possible adverse situations in between or beyond the one-year horizon. Ruin theory precisely accounts for the insured risk during the whole life-time of the business or until any given time-horizons. This is why practitioners often look at risks in the ruin context when building internal models. Risk measures derived from ruin theory often provide more robust risk indicators.

Dhaene et al. (2003) give ample motivation for an exponential risk measure inherited from the Cramèr–Lundberg upper bound for the ruin probability in a discrete-time ruin model. Cheridito et al. (2006) mention (in an application of their study on coherent risk measures for unbounded stochastic processes) a VaR-type risk measure based on the infinite-time ruin probability itself. Overall, over the last few years, the relative position and relation between risk measures that fulfill a list of axioms on the one hand, and classical ruin theory on the other hand, has often been a matter of debate. Trufin et al. (2011) take up this issue and investigate in more detail some properties of the VaR-type risk measure based on the ultimate ruin probability that is mentioned in Cheridito et al. (2006).

Over the years, the management of the liquidity risk has become a major concern for the insurance industry. The recent financial crisis tells us how this risk can be devastating for a financial institution. In this paper, we propose to study a risk measure derived from ruin theory that takes into account liquidity risk of an insurance business over a given period of time.

In risk theory, many risk measures have been considered for the classical continuous-time risk model. In addition to the well-known finite-time and infinite-time ruin probabilities, some others risk measures have been deeply investigated throughout numerous articles (see Gerber (1988), Dufresne and Gerber (1988) and Picard (1994) for instance). Let us mention the time to ruin, the severity of ruin, the time spent below zero from the first ruin to the first time of recovery, the maximal ruin severity or also the aggregate severity of ruin until recovery. The total time spent below zero has also been studied by Dos Reis (1993), using some results of Gerber (1988). However, all those risk measures are either drawn from the infinite-time ruin theory or involve the behavior of the risk process between ruin times and recovery times. This is

* Corresponding author. Tel.: +1 4185766973.
E-mail addresses: stephane.loisel@univ-lyon1.fr (S. Loisel), julien.trufin@act.ulaval.ca (J. Trufin).

http://dx.doi.org/10.1016/j.insmatheco.2014.01.012
0167-6687/

why Loisel (2005) introduces a risk measure based on a fixed time interval, finite or infinite, i.e. the expected time-integrated negative part of general risk processes on a fixed time interval $[0, T]$, also called the expected area in red. This risk measure, if multiplied by a constant interest rate, can be interpreted as the total cost of debt financing during the period $[0, T]$. Also, it can be regarded as a natural time extension of the concepts of Tail-VaR and severity at ruin in a dynamic context and is less binary than the ruin probability. The author derives some expressions for this new risk measure and obtains a closed-form formula in the compound Poisson risk model in infinite-time for exponential claims. Biard et al. (2010) investigate the asymptotic behavior of the expected area in red and discuss an optimal allocation problem with two business lines.

The expected area in red appears to be a relevant risk indicator for quantifying the liquidity risk of an insurance business over a given time horizon. In practice, when the surplus of an insurance business gets below some level, the insurer needs to finance some kind of debt and get supported by fresh money from the parent company or from another entity. Of course, this help cannot last for too long or concern a too high amount. So, if the area in red is too large, the insurer is likely to fail to pay its liabilities in the short term since it will not be able to get help to refinance. In this paper, we adopt the expected area in red to reflect the liquidity risk of an insurer and we look at the smallest capital that ensures it is less than a given level. While the ruin probability focuses on the risk that (at least) one liquidity issue occurs over a finite- or infinite-time horizon, we are here more interested in what happens once the insurer encounters liquidity problems and in controlling their potential impact. Note that in the sequel, the zero surplus does not necessarily correspond to economic ruin. It may correspond to a risk tolerance level, or to a level after which some controls or penalties would affect business and profitability.

The purpose of this paper is to consider certain properties of a risk measure that is derived from the expected area in red. It provides tools to enable a better assessment of the riskiness (through the perspective of the liquidity risk) of certain financial positions in the insurance context. It also makes an interesting link between the initial reserve optimal allocation problem and a new problem of risk budget allocation, that we define and solve. More generally, this paper will contribute to improve the understanding of the connections between the axiomatic framework of risk measures and ruin theory.

Our paper is organized as follows. In Section 2, we set up the scene by presenting the risk process we are dealing with in the present analysis and by recalling the definitions of some stochastic orders used in the following. Next, in Section 3, we introduce the studied risk measure and investigate some of its properties. Further, numerical illustrations are performed within the compound Poisson model for an infinite horizon to illustrate the diversification benefit resulting from the aggregation of two business lines. Finally, Section 4 determines the way(s) to allocate a risk limit set at a company level to the subsequent business lines in order to minimize the overall capital needs.

2. Risk model and stochastic orders

2.1. The model

The surplus process (or risk process) is defined as

$$U_t = u + c\,t - S_t, \quad t \geq 0,$$

where U_t is the insurer's capital at time t starting from some initial capital $U_0 = u$, c is the (constant) premium income per unit of time and $S_t = \sum_{k=1}^{N_t} X_k$ is the total claim amount up to time t, with N_t the corresponding number of claims, and X_k the size of the kth claim, assumed to be non-negative. The claim number process $\mathbf{N} = \{N_t,\ t \geq 0\}$ is governed by a sequence of identically distributed inter-occurrence times T_k with a common distribution function F_T. The X_k's are identically distributed as X, with distribution function F_X. Let us notice that we do not require the independence of the inter-occurrence times T_k's nor of the claim sizes X_k's. Also, we do not assume the X_k's to be independent from the T_k's.

The expected area in red on a fixed time interval $[0, T]$ is defined as

$$\mathbb{E}\left[I_{T,c}(u)\right] = \mathbb{E}\left[\int_{t=0}^{T} |U_t| 1_{\{U_t<0\}} dt\right].$$

It can also be expressed as follows, which will be useful in our analysis:

$$\begin{aligned}\mathbb{E}\left[I_{T,c}(u)\right] &= \mathbb{E}\left[\int_{t=0}^{T} |U_t| 1_{\{U_t<0\}} dt\right] \\ &= \int_{t=0}^{T} \mathbb{E}\left[|U_t| 1_{\{U_t<0\}}\right] dt \quad \text{using Fubini's theorem} \\ &= \int_{t=0}^{T} \mathbb{E}\left[(S_t - c\,t - u)_+\right] dt. \qquad (2.1)\end{aligned}$$

Henceforth, when needed to make explicit the dependence on $\mathbf{S} = \{S_t,\ t \geq 0\}$, we will denote $\mathbb{E}\left[I_{T,c}(u)\right]$ as $\mathbb{E}\left[I^{(\mathbf{S})}_{T,c}(u)\right]$.

2.2. Stochastic orders

In this section, we recall the definitions of some stochastic orders that will be useful in the following. For more details, we refer the interested reader, e.g., to Denuit et al. (2005).

Given two random variables X and Y, X precedes Y in the usual stochastic order, denoted as $X \preceq_{\text{st}} Y$, if

$$\overline{F}_X(u) \leq \overline{F}_Y(u) \quad \text{for all } u,$$

where $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$. The latter is also equivalent to the inequality $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ for any non-decreasing function g such that the expectations exist.

The usual stochastic order compares the sizes of the risks. The convex order focuses on their variabilities. It allows us to compare two risks with identical means. Given two random variables X and Y such that $\mathbb{E}[X] = \mathbb{E}[Y]$, X precedes Y in the convex order, denoted as $X \preceq_{\text{cx}} Y$, when

$$\int_t^{\infty} \overline{F}_X(u)\,du \leq \int_t^{\infty} \overline{F}_Y(u)\,du \quad \text{for all } t. \qquad (2.2)$$

The inequality in (2.2) can be equivalently written as

$$\mathbb{E}[(X-t)_+] \leq \mathbb{E}[(Y-t)_+] \quad \text{for all } t. \qquad (2.3)$$

From (2.3) it follows that $X \preceq_{\text{cx}} Y$ if, and only if, $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ for all convex functions $g : \mathbb{R}^+ \to \mathbb{R}$, provided that the expectations exist.

The increasing convex, or stop loss, order combines the aspects of size (as $\preceq_{\text{st}}$) and variability (as $\preceq_{\text{cx}}$). Being smaller in the stop-loss order means "smaller" and "less variable". By definition, X is said to be smaller than Y in the stop-loss order, denoted as $X \preceq_{\text{icx}} Y$, when

$$\int_t^{\infty} \overline{F}_X(u)\,du \leq \int_t^{\infty} \overline{F}_Y(u)\,du \quad \text{for all } t,$$

or equivalently if

$$\mathbb{E}[(X-t)_+] \leq \mathbb{E}[(Y-t)_+] \quad \text{for all } t. \qquad (2.4)$$