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Polynomial extensions of distributions and their applications in actuarial and financial modeling

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HIGHLIGHTS

- Construction of distributions and orthogonal polynomials via Pearson's equation.
- Modeling returns and claims by means of polynomially extended distributions.
- Calculation of option prices, stop-loss premiums and probabilities of ruin.

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ABSTRACT

The paper deals with orthogonal polynomials as a useful technique which can be attracted to actuarial and financial modeling. We use Pearson's differential equation as a way for orthogonal polynomials construction and solution. The generalized Rodrigues formula is used for this goal. Deriving the weight function of the differential equation, we use it as a basic distribution density of variables like financial asset returns or insurance claim sizes. In this general setting, we derive explicit formulas for option prices as well as for insurance premiums. The numerical analysis shows that our new models provide a better fit than some previous actuarial and financial models.

1. Introduction

Probabilistic models are widely used in mathematical finance and actuarial sciences. For the reason of simplicity, the most popular models in these areas usually start with many assumptions and restrictions, which lead to an ideal situation and end up with some fixed distribution for estimated financial and actuarial assets. The Black–Scholes model has become the most well-known model in the analysis of financial asset pricing, with the benefit of its properties in mathematical theory and simplicity in numerical realization. But the shortcomings of the Black–Scholes model are obvious—its assumptions on the trading market are too ideal to be possible in the real world (see Black, 1989). Normal distribution for the logarithmic returns of financial assets is one of the most important implications of the Black–Scholes model, and this result is severely doubted by other empirical and theoretical studies. Implications of the normality of financial asset returns, that disagree with the historical data, include the following: extreme returns (more than 3 standard deviations from mean) are only assigned a tiny probability which is nearly neglectable. Financial returns must have the same probabilities for the same level of prosperity and recession, as a result of the symmetry of the normal distribution. Also, financial returns must have fixed skewness and kurtosis. Other distributions are needed for better estimating the financial asset returns.

One suitable extension of the Black–Scholes model is the Gram–Charlier model. The Gram–Charlier model uses the product of the normal density and a 4th degree polynomial as the financial return density, and thus allows arbitrary skewness and kurtosis (Madan and Milne, 1994). This is the first example of using polynomial to generate normal-like distributions to model financial returns. The polynomial used in the Gram–Charlier model is a linear combination of the Hermite polynomial series, which is an orthogonal polynomial series based on the normal distribution (see Fedoryuk, 2001). The Hermite polynomial series is a special case of polynomial solutions for the so-called Pearson's differential equation. Hence, aiming to extend this approach, we should concentrate first on Pearson's differential equation which gives us a base for such research.

The general polynomial solutions for Pearson's differential equation are solved by the generalized Rodrigues formula (see Raposo et al., 2007). Weight functions can be generated from different differential equations, as a base of the inner product on the space

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of polynomials, and orthogonal polynomials based on this inner product can be derived. The leading idea of our approach is to standardize the weight function and further use this function for the basic density function construction. Using the product of this basic density function and a linear combination of the corresponding orthogonal polynomials to fit the density function of the variables of interest, like logarithmic financial returns or insurance portfolio claims. This paper is devoted to the mathematical properties of polynomials as the solution of Pearson's differential equation, and the model extensions of the above approach applied in financial and actuarial sciences. To the best of our knowledge, such an approach is not very well developed yet in these areas.

The paper is organized as follows. Section 2 is devoted to the discussion of Pearson's differential equation and its polynomial solutions given by the generalized Rodrigues formula. Classification and related properties of the polynomial solutions are also discussed. Section 3 is devoted to the financial application generated by this approach. Section 4 is devoted to the applications generated by this approach in actuarial sciences. Section 5 is devoted to the empirical studies of the above applications in financial and actuarial sciences.

2. The polynomial solutions of Pearson's differential equation

This section gives a review of the famous Pearson's differential equation and shows how to use the generalized Rodrigues formula to solve it. The Rodrigues formula was first introduced independently by Rodrigues, Ivory and Jacobi, to provide a construction of the Legendre polynomials (see Askey, 2005). Later such an equation was exploited in more general aspects. Many properties of the polynomials can be recognized using the system of the differential equation and the generalized Rodrigues formula. These properties are extremely important and useful to provide financial and actuarial model extensions.

Pearson's differential equation is defined as follows:

$$s_2(x)F''(x) + s_1(x)F'(x) + \lambda F(x) = 0, \tag{1}$$

where $s_1(x)$ and $s_2(x)$ are polynomials of x with at most first and second degrees. The general solution of this differential equation can be expressed as a generalized hypergeometric function. When the parameters of the differential equation satisfy to a certain condition, the solution is reduced to a polynomial. Suppose a polynomial $F(x)$ of degree n is a solution of Eq. (1), the following equation is obtained by eliminating the term x^n on the left hand side of the differential equation (see Raposo et al., 2007)

$$\lambda = -ns_1'(x) - \frac{n(n-1)s_2''(x)}{2}. \tag{2}$$

Furthermore, one can prove that under condition (2), there exists a unique polynomial solution with degree n satisfying the differential equation, if we neglect the effect of the scalar multiplication. Denote this unique solution as $F_n(x)$.

The second property of the polynomial series begins from differentiation. Calculating the mth derivative of Eq. (1) with the polynomial $F_n(x)$, we obtain the following equation:

$$s_2(x)F_n^{(m+2)}(x) + (s_1(x) + ms_2'(x))F_n^{(m+1)}(x) + \left(\frac{m(m-1)}{2}s_2''(x) + ms_1'(x) + \lambda\right)F_n^{(m)}(x) = 0.$$

If we denote

$$s_{1,m}(x) = s_1(x) + ms_2'(x), \tag{3}$$

$$\lambda_{n,m} = \frac{m(m-1) - n(n-1)}{2}s_2''(x) + (m-n)s_1'(x), \tag{4}$$

then the above differential equation becomes

$$s_2(x)(F_n^{(m)}(x))'' + s_{1,m}(x)(F_n^{(m)}(x))' + \lambda_{n,m}F_n^{(m)}(x) = 0. \tag{5}$$

Thus, $F_n^{(m)}(x)$, as a polynomial of degree $n-m$, is the solution of a similar differential equation. Let us introduce the weight function $w(x)$, as a solution of the differential equation (under a proper scale):

$$(s_2(x)w(x))' = s_1(x)w(x). \tag{6}$$

It is called as the weight function because it plays the role of a weight function in the inner product and orthogonality between polynomials. The solution of the above equation can be obtained as (see Raposo et al., 2007)

$$w(x) = A\frac{1}{s_2(x)}\exp\left(\int\frac{s_1(x)}{s_2(x)}dx\right). \tag{7}$$

The weight function is a non-polynomial solution of Pearson's differential equation connected to Pearson's differential equation. It is clear from the following observation. By differentiating (6), we arrive to the next equation

$$s_2(x)w''(x) + (2s_2'(x) - s_1(x))w'(x) + (s_2''(x) - s_1'(x))w(x) = 0.$$

The derivatives of the solution $F_n(x)$ are given by the generalized Rodrigues formula presented in the following theorem.

Theorem 2.1. *If $F_n(x)$ is an nth-degree polynomial and the solution of the differential equation (1), the mth derivative of $F_n(x)$ should follow the formula below:*

$$F_n^{(m)}(x) = N_{n,m}\frac{1}{w(x)s_2^m(x)}\frac{d^{n-m}}{dx^{n-m}}(w(x)s_2^n(x)), \tag{8}$$

$$\text{where } N_{n,m} = (-1)^m N_{n0}\prod_{k=0}^{m-1}\lambda_{n,k},$$

$$\text{for any } n \geq m \geq 0,\ N_{n,0} \in \mathbf{R}. \tag{9}$$

A brief proof of Theorem 2.1 is given in Raposo et al. (2007). We give the full proof of the formulas (8)–(9) by the induction method.

Proof. Let $F_n^*(x) = w(x)s_2^n(x)$. By calculating the first and second derivatives of $F^*(x)$, we get the following formulas:

$$F_n^{*'}(x) = w(x)s_2^{n-1}(x)(s_1(x) + (n-1)s_2'(x)), \tag{10}$$

$$\begin{aligned}F_n^{*''}(x) \;=\; & w(x)s_2^{n-2}(x)(s_1(x) + (n-2)s_2'(x))(s_1(x) \\ & + (n-1)s_2'(x)) + w(x)s_2^{n-1}(x)(s_1'(x) \\ & + (n-1)s_2''(x)).\end{aligned} \tag{11}$$

From the above formulas, we can verify that

$$s_2(x)F_n^{*''}(x) - (s_1(x) + (n-2)s_2'(x))F^{*'}(x) - (s_1'(x) + (n-1)s_2''(x))F^*(x) = 0. \tag{12}$$

Differentiating the above formula $n-m-2$ times, we have

$$\begin{aligned}& s_2(x)\frac{d^{n-m}}{dx^{n-m}}F_n^*(x) - (s_1(x) + ms_2'(x))\frac{d^{n-m-1}}{dx^{n-m-1}}F_n^*(x) \\ & + \left(\left(\frac{m(m+1)}{2} - \frac{n(n-1)}{2}\right)s_2''(x) \right. \\ & \left. + (m-n+1)s_1'(x)\right)\frac{d^{n-m-2}}{dx^{n-m-2}}F_n^*(x) = 0.\end{aligned} \tag{13}$$

To prove the theorem, we use the induction method. When $m = n$, $F_n^{(m)}$ is a constant, as the result of differentiation with a