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On multivariate extensions of Conditional-Tail-Expectation

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ABSTRACT

In this paper, we introduce two alternative extensions of the classical univariate *Conditional-Tail-Expectation* (CTE) in a multivariate setting. The two proposed multivariate CTEs are vector-valued measures with the same dimension as the underlying risk portfolio. As for the multivariate Value-at-Risk measures introduced by Cousin and Di Bernardino (2013), the *lower-orthant CTE* (*resp.* the *upper-orthant CTE*) is constructed from level sets of multivariate distribution functions (*resp.* of multivariate survival distribution functions). Contrary to allocation measures or systemic risk measures, these measures are also suitable for multivariate risk problems where risks are heterogeneous in nature and cannot be aggregated together. Several properties have been derived. In particular, we show that the proposed multivariate CTE-s satisfy natural extensions of the positive homogeneity property, the translation invariance property and the comonotonic additivity property. Comparison between univariate risk measures and components of multivariate CTE is provided. We also analyze how these measures are impacted by a change in marginal distributions, by a change in dependence structure and by a change in risk level. Sub-additivity of the proposed multivariate CTE-s is provided under the assumption that all components of the random vectors are independent. Illustrations are given in the class of Archimedean copulas.

1. Introduction

As illustrated by the recent financial turmoil, risks are strongly interconnected. Consequently, risk quantification in multivariate settings has recently been the subject of great interest. Much research has been devoted to construction of risk measures that account both for marginal effects and dependence between risks.

In the literature, several generalizations of the classical univariate Conditional-Tail-Expectation (CTE) have been proposed, mainly using as conditioning events the total risk or some extreme risks. These measures can be used as capital allocation rules for financial institutions. The aim is to find the contribution of each subsidiary (or risk category) to the total economic capital. As can be seen in Scaillet (2004) and Tasche (2008), the Euler or Shapley–Aumann allocation rule associated with a particular univariate risk measure (such as VaR or CTE) involves the dependence structure between marginal and aggregated risks. More formally, let $\mathbf{X} = (X_1, \dots, X_d)$ represent the risk exposures of a given financial institution, where, for any $i = 1, \dots, d$, the component X_i denotes the marginal risk (usually claim or loss) associated with the underlying entity i (the latter could be, for instance, a subsidiary, an operational branch or a risk category). Then, the sum $S = X_1 + \cdots + X_d$ corresponds to the company aggregated risk, whereas $X_{(1)} = \min\{X_1, \dots, X_d\}$ and $X_{(d)} = \max\{X_1, \dots, X_d\}$ are the extreme risks. In capital allocation problems, we are not only interested in the "stand-alone" risk measures $\mathrm{CTE}_\alpha(X_i) = \mathbb{E}[X_i \mid X_i > Q_{X_i}(\alpha)]$, where $Q_{X_i}(\alpha) = \inf\{x \in \mathbb{R}_+ : F_{X_i}(x) \geq \alpha\}$ is the univariate quantile function of X_i at risk level α, but also in

$$\mathrm{CTE}_\alpha^{\mathrm{sum}}(X_i) = \mathbb{E}[X_i \mid S > Q_S(\alpha)], \tag{1}$$

$$\mathrm{CTE}_\alpha^{\mathrm{min}}(X_i) = \mathbb{E}[X_i \mid X_{(1)} > Q_{X_{(1)}}(\alpha)], \tag{2}$$

$$\mathrm{CTE}_\alpha^{\mathrm{max}}(X_i) = \mathbb{E}[X_i \mid X_{(d)} > Q_{X_{(d)}}(\alpha)], \tag{3}$$

for $i = 1, \dots, d$. The interested reader is referred to Cai and Li (2005) for further details. An explicit formula for $\mathrm{CTE}_\alpha^{\mathrm{sum}}(X_i)$ is provided in Landsman and Valdez (2003) in the case of elliptic distribution functions, Cai and Li (2005) for *phase-type* distributions and in Bargès et al. (2009) for Farlie–Gumbel–Morgenstern family of copulas. Furthermore, we recall that $\mathrm{CTE}_\alpha^{\mathrm{sum}}(X_i)$ corresponds to the "Euler allocation rule" associated with the univariate CTE (see, e.g., Tasche, 2008).

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Another problem which recently receives a great interest is the construction of systemic risk measures. One of the proposed measure is the Marginal Expected Shortfall (MES) defined as the expected loss on its equity return (X) conditional on the occurrence of a loss in the aggregated return of the financial sector (Y), i.e.,

$$\mathrm{MES}_\alpha(X) = \mathbb{E}[X \mid Y > Q_Y(\alpha)], \tag{4}$$

where $Q_Y(\alpha)$ is the (α)-th quantile of the distribution of Y. The MES of a financial institution aims at detecting which firms in the economy are the more vulnerable in case of a global financial distress. On mathematical grounds, this measure is similar to the allocation measure $\mathrm{CTE}^{\mathrm{sum}}$. The interested reader is referred to Acharya et al. (2010) or Brownlees and Engle (2012) for more details. Cai et al. (2013) propose a non-parametric estimator of the MES using extreme value theory. The CoVaR (conditional VaR) of company i is instead given by

$$\mathrm{CoVaR}^i_\alpha(X) = \mathrm{VaR}_\alpha\left(S \mid X \geq \mathrm{VaR}_\alpha(X)\right). \tag{5}$$

As opposed to the MES, the CoVaR (conditional VaR) is constructed in order to identify which firms in the economy have a great importance in terms of systemic risk (see Adrian and Brunnermeier, 2011 or Mainik and Schaanning, 2012).

However, the previous risk measures are not suitable for multivariate risk problem where risks are heterogeneous in nature and thus cannot be aggregated together or even compared. This is the case for instance for risks which are difficult to expressed under the same numéraire or when one has to deal with non-monetary risks or exogenous risks. The literature which deals with risk measures for intrinsically multivariate problems can be divided in two categories. The first group of papers are interested in extending classical univariate axioms to different multivariate settings (see for instance Jouini et al., 2004, Burgert and Rüschendorf, 2006, Rüschendorf, 2006, Cascos and Molchanov, 2007, Hamel and Heyde, 2010 and Ekeland et al., 2012). One of the objectives is to derive theoretical representation of risk measures. This is done without proposing tractable constructions for the axiom-consistent multivariate measures. Another group of papers investigates different generalizations of the concept of quantiles in a multivariate setting. Unsurprisingly, the main difficulty regarding multivariate generalizations of quantile-based risk measures (as the VaR and the CTE) is the fact that vector preorders are, in general, partial preorders. Then, what can be considered in a context of multidimensional portfolios as the analogous of a "worst case" scenario and a related "tail distribution"? For example, Massé and Theodorescu (1994) define multivariate α-quantiles for bivariate distribution as the intersection of half-planes whose the distribution is at least equal to α. Koltchinskii (1997) provides a general treatment of multivariate quantiles as inversions of mappings. Another approach is to use geometric quantiles (see, for example, Chaouch et al., 2009). Along with the geometric quantile, the notion of depth function has been developed in recent years to characterize the quantile of multidimensional distribution functions (for further details see, for instance, Chauvigny et al., 2011). We refer to Serfling (2002) for a review of multivariate quantiles.

When it turns to generalize the Value-at-Risk measure, Embrechts and Puccetti (2006), Nappo and Spizzichino (2009), Prékopa (2012) use the notion of quantile curve but these papers do not investigate whether these measures are compatible with some desirable axioms. Moreover, the proposed risk measures are hyperspaces and thus quantify a vector of risks with an infinite number of points. Contrarily to the latter approach, the multivariate Conditional-Tail-Expectation proposed in this paper quantifies multivariate risks in a more parsimonious and synthetic way. This feature can be relevant for operational applications since it can ease discrimination between portfolio of risks. Lee and Prékopa (2013) introduce a real-valued measure of multivariate risks which also bears on quantile curves but the proposed measure relies on a somehow arbitrary convex combination.

We propose two vector-valued extensions of the univariate Conditional-Tail-Expectation. The lower-orthant CTE of a random vector X (introduced by Di Bernardino et al., 2013 in a bivariate setting) is defined as the conditional expectation of X given that the latter is located in the α-upper level set of its distribution function. The upper-orthant CTE of X is defined as the conditional expectation of X given that the latter is in the $(1-\alpha)$-lower level set of its survival function. Several properties have been derived. We provide an integral representation of the proposed measures in terms of the multivariate VaR introduced in Cousin and Di Bernardino (2013) and we show that the proposed multivariate CTE-s satisfy natural extensions of the positive homogeneity property, the translation invariance property and the comonotonic additivity property. We show that the proposed measures are sub-additive for independent vectors with independent components. We also provide comparisons between univariate risk measures and components of the proposed multivariate CTE. We analyze how these measures are impacted by a change in marginal distributions, by a change in dependence structure and by a change in risk level.

The paper is organized as follows. In Section 2, we give the definition of the lower-orthant and the upper-orthant Conditional-Tail-Expectation measures. We then show that these measures satisfy multivariate extensions of Artzner et al. (1999)'s invariance properties (see Section 2.1). Illustrations in some Archimedean copula cases are presented in Section 2.2. We also compare the components of these multivariate CTE measures with the associated univariate VaR, the associated univariate CTE and with the multivariate lower-orthant and upper-orthant VaR previously introduced by Cousin and Di Bernardino (2013) (see Section 2.3). The behavior of our CTE-s with respect to a change in marginal distributions, a change in dependence structure and a change in risk level α is discussed respectively in Sections 2.4–2.6. The conclusion discusses open problems and possible directions for future work.

2. Multivariate generalization of the Conditional-Tail-Expectation measure

As in the univariate case, the multivariate VaR introduced in Cousin and Di Bernardino (2013) does not give any information regarding the upper tail of the loss distribution function and especially its degree of thickness above the VaR threshold. In an univariate setting, the problem has been overcome by considering for instance the Conditional-Tail-Expectation (CTE) risk measure,[2] which is defined as the conditional expectation of losses given that the latter exceed VaR. Following Artzner et al. (1999), the CTE at level α for a distribution function F (or survival function $\overline{F}$) is given by

$$\mathrm{CTE}_\alpha(X) := \mathbb{E}[X \mid X \geq \mathrm{VaR}_\alpha(X)], \tag{6}$$

where $\mathrm{VaR}_\alpha(X)$ is the univariate Value-at-Risk defined by

$$\begin{aligned}\mathrm{VaR}_\alpha(X) &:= \inf\{x \in \mathbb{R} : F(x) \geq \alpha\} \\ &= \inf\left\{x \in \mathbb{R} : \overline{F}(x) \leq 1-\alpha\right\}.\end{aligned}$$

Since the sets $\{X \geq \mathrm{VaR}_\alpha(X)\}$, $\{F(X) \geq \alpha\}$ and $\{\overline{F}(X) \leq 1-\alpha\}$ correspond to the same event in a univariate setting, the CTE can alternatively be defined[3] as

$$\mathrm{CTE}_\alpha(X) := \mathbb{E}[X \mid F(X) \geq \alpha] = \mathbb{E}[X \mid \overline{F}(X) \leq 1-\alpha]. \tag{7}$$

The CTE can then be viewed as the conditional expectation of X given that X falls into the α-lower-level set of its distribution function $\underline{L}(\alpha) := \{x \in \mathbb{R}_+ : F(x) \geq \alpha\}$ or equivalently in the $(1-\alpha)$-upper-level set of its survival function $\overline{L}(\alpha) := \{x \in \mathbb{R}_+ : \overline{F}(x) \leq$

2 This measure is also called Tail Conditional Expectation. As far as continuous distribution functions are considered, the CTE measure is coherent in the sense of Artzner's axioms and it coincides with the Expected Shortfall or Tail VaR.

3 Note that this definition does not depend on VaR.