Insurance: Mathematics and Economics 59 (2014) 1–10

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

# On the moments of the time to ruin in dependent Sparre Andersen models with emphasis on Coxian interclaim times

Wing Yan Lee *, Gordon E. Willmot

*Department of Statistics and Actuarial Science, University of Waterloo, 200 University Avenue West, Waterloo, Canada*

ARTICLE INFO

*Article history:*
Received April 2014
Received in revised form
August 2014
Accepted 7 August 2014

*Keywords:*
Time to ruin
Moment
Defective renewal equation
Laplace transform
Coxian interclaim time

ABSTRACT

The structural properties of the moments of the time to ruin are studied in dependent Sparre Andersen models. The moments of the time to ruin may be viewed as generalized versions of the Gerber–Shiu function. It is shown that structural properties of the Gerber–Shiu function hold also for the moments of the time to ruin. In particular, the moments continue to satisfy defective renewal equations. These properties are discussed in detail in the model of Willmot and Woo (2012), which has Coxian interclaim times and arbitrary time-dependent claim sizes. Structural quantities needed to determine the moments of the time to ruin are specified under this model. Numerical examples illustrating the methodology are presented.

© 2014 Elsevier B.V. All rights reserved.

## 1. Introduction and background

Consider the insurance surplus process $\{U_t, t \geq 0\}$ defined by

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i,$$

where $u$ ($u \geq 0$) is the initial surplus and $c$ is the premium rate in one unit of time. $\{N_t, t \geq 0\}$ is a claim number process which is defined through a sequence of independent and identically distributed (i.i.d.) interclaim time random variables $\{V_i, i = 1, 2, \ldots\}$, where $V_1$ is the time until first claim and $V_i$ is the time between $(i-1)$th and $i$th claim for $i = 2, 3, \ldots$. Let us denote the marginal probability density function (pdf) and cumulative distribution function (cdf) of the interclaim time $V$ be $k(t)$ and $K(t)$ respectively, where $V$ is any arbitrary $V_i$.

Moreover, $\{Y_i, i = 1, 2, \ldots\}$ is assumed to be an i.i.d. sequence of claim size random variables. The dependent Sparre Andersen model is considered in this paper, where the pairs $\{(V_i, Y_i), i = 1, 2, \ldots\}$ are i.i.d., but $V_i$ and $Y_i$ may be dependent. Let $f(t, y)$ be the joint pdf of the pair $(V, Y)$ when $V = t$ and $Y = y$. Finally, let us assume that the positive security loading condition $E[cV] > E[Y]$ holds in this paper.

Let $T$ be the time to ruin for the process $\{U_t, t \geq 0\}$, which is defined by $T = \inf\{t \geq 0 : U_t < 0\}$ and $T = \infty$ if $U_t$ is nonnegative for all $t \geq 0$. In this paper, we consider the generalized Gerber–Shiu function introduced in Cheung et al. (2010), i.e.

$$m_\delta(u) = E[e^{-\delta T} w(U_{T-}, |U_T|, R_{N_T - 1}) I(T < \infty) | U_0 = u], \tag{1.1}$$

where $\delta \geq 0$ and the penalty function $w$ satisfies mild integrability conditions and $I(A)$ is an indicator function which takes value 1 if the event $A$ occurs and 0 otherwise. The random variables $U_{T-}$ and $|U_T|$ represent the surplus before ruin and the deficit at ruin respectively. $N_T$ denotes the number of claims until ruin. $R_n$ is defined as $R_0 = u$ and $R_n = u + \sum_{i=1}^{n}(cV_i - Y_i)$ for $n = 1, 2, \ldots$. Therefore, $R_{N_T-1}$ is equal to $u$ if ruin occurs on the first claim ($N_T = 1$). For ruin occurs on claim subsequent to the first ($N_T > 1$), $R_{N_T-1}$ is the surplus immediately after the second last claim before ruin.

The time to ruin is one of the central quantities in risk theory and its moments have been studied by many authors. In the classical Poisson risk model, Lin and Willmot (2000) showed that the moments of the time to ruin satisfy a sequence of defective renewal equations. Based on this result, Drekic and Willmot (2005) evaluated the moments by assuming phase-type claim sizes. There are also approximation results from the literature. Egidio dos Reis (2000) considered the moments of the time to ruin in a discrete time compound Poisson risk model and used the result for approximation in the continuous case. Dickson and Waters (2002) approximated the moments by using the results in Lin and Willmot (2000). In Pitts and Politis (2008), the moments were approximated with a functional approach and numerical examples were provided for the expected time to ruin.

Apart from the classical Poisson model, the moments of the time to ruin have also been considered in more general risk models. Dickson and Hipp (2001) studied the moments in a Sparre

* Corresponding author. Tel.: +1 2269781353.
*E-mail address:* wy8lee@uwaterloo.ca (W.Y. Lee).

http://dx.doi.org/10.1016/j.insmatheco.2014.08.003
0167-6687/© 2014 Elsevier B.V. All rights reserved.

Andersen model with Erlang (2) interclaim times. Yu et al. (2010) assumed a Markovian risk model and determined the moments of the time to ruin by matrix analysis. Li and Lu (2013) considered a surplus process with interest and the time to ruin was analyzed using stopping time arguments.

In this paper, the (discounted) generalized $k$th moment of the time to ruin to be considered is defined by

$$m_{k,\delta}(u) = E[T^k e^{-\delta T} w(U_{T-}, |U_T|, R_{N_T-1}) I(T < \infty)|U_0 = u] \quad (1.2)$$

for $k = 0, 1, 2, \ldots$. (1.2) may be viewed as a generalized version of the Gerber–Shiu function (1.1), recovered with $k = 0$. Furthermore, structural properties of the Gerber–Shiu function will be shown to hold also for (1.2). In Section 2, structural properties of the moments of the time to ruin are analyzed in an arbitrary dependent Sparre Andersen model. A set of defective renewal equations which is recursive in $k$ will be given for (1.2). In Section 3, the fairly general model of Willmot and Woo (2012) is considered which assumes that the interclaim times are Coxian and the claim sizes are time-dependent. It is a generalization of the independent model considered by Li and Garrido (2005) and Willmot and Woo (2010). Structural quantities needed to solve for the moments of the time to ruin are specified under this model. Section 4 presents two numerical examples illustrating the accuracy of the approach as well as the generality of its applicability.

For notational convenience, let us denote the Laplace transform of an arbitrary non-negative function $h(\cdot)$ by $\tilde{h}(s) = \int_0^\infty e^{-sx} h(x)dx$ in this paper. Also, note that $\sum_{j=i}^k = 0$ for $i > k$.

## 2. Properties of the moments of time to ruin

The Gerber–Shiu functions introduced in Section 1 have been shown to satisfy defective renewal equations in many papers. For example, readers can refer to Gerber and Shiu (1998) and Cheung et al. (2010). Based on these references, a brief description of the argument is given in the following.

Consider an arbitrary dependent Sparre Andersen model. The joint density of $T$, $U_{T-}$, $|U_T|$ and $R_{N_T-1}$ is first defined. Given initial surplus $u$ and for ruin occurred on the first claim, let $h_1(x, y|u)$ be the joint defective density of the surplus before ruin ($x$) and the deficit at ruin ($y$). Since ruin is on the first claim, the time of ruin ($t$) is given by $t = (x - u)/c$ and hence

$$h_1(x, y|u) = \begin{cases} \frac{1}{c} f\left(\frac{x-u}{c}, x+y\right), & x > u \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

Also, by definition, $R_{N_T-1} = u$ if ruin is on the first claim. Given initial surplus $u$ and for ruin on claims subsequent to the first, let

$$h_2(t, x, y, v|u) \quad (2.2)$$

where $v < x$, be the joint defective density of the time of ruin ($t$), the surplus before ruin ($x$), the deficit at ruin ($y$) and the surplus immediately after the second last claim before ruin ($v$).

Then, define the discounted densities

$$h_{1,\delta}(x, y|u) = e^{-\delta\left(\frac{x-u}{c}\right)} h_1(x, y|u), \quad (2.3)$$

$$h_{2,\delta}(x, y, v|u) = \int_0^\infty e^{-\delta t} h_2(t, x, y, v|u)dt \quad (2.4)$$

and

$$h_\delta(x, y|u) = h_{1,\delta}(x, y|u) + \int_0^x h_{2,\delta}(x, y, v|u)dv. \quad (2.5)$$

Cheung et al. (2010) showed that (1.1) satisfy the defective renewal equation

$$m_\delta(u) = \phi_\delta \int_0^u m_\delta(u - y) f_\delta(y)dy + v_\delta(u), \quad (2.6)$$

where $f_\delta(y)$ is the discounted ladder height density defined by

$$f_\delta(y) = \frac{1}{\phi_\delta} \int_0^\infty h_\delta(x, y|0)dx \quad (2.7)$$

with $\phi_\delta = \int_0^\infty \int_0^\infty h_\delta(x, y|0)dxdy$, and

$$\begin{aligned} v_\delta(u) = &\int_u^\infty \int_0^\infty w(x + u, y - u, u) h_{1,\delta}(x, y|0)dxdy \\ &+ \int_u^\infty \int_0^\infty \int_0^x w(x + u, y - u, v + u) \\ &\times h_{2,\delta}(x, y, v|0)dvdxdy. \end{aligned} \quad (2.8)$$

This result is now generalized to the moments of the time to ruin. For notational convenience, define

$$h^{*k}_{1,\delta}(x, y|u) = \left(\frac{x-u}{c}\right)^k h_{1,\delta}(x, y|u), \quad (2.9)$$

$$h^{*k}_{2,\delta}(x, y, v|u) = \int_0^\infty t^k e^{-\delta t} h_2(t, x, y, v|u)dt \quad (2.10)$$

and

$$h^{*k}_\delta(x, y|u) = h^{*k}_{1,\delta}(x, y|u) + \int_0^x h^{*k}_{2,\delta}(x, y, v|u)dv \quad (2.11)$$

for $k = 0, 1, 2, \ldots$. In fact, (2.9)–(2.11) are functions related to (2.3)–(2.5) respectively by a $k$th order differentiation.

**Theorem 2.1.** *Consider the dependent Sparre Andersen model as described in Section 1 with initial surplus u. The generalized kth moment of the time to ruin, i.e.* $m_{k,\delta}(u)$ *defined in (1.2), satisfies a defective renewal equation. For* $k = 0, 1, 2, \ldots$,

$$m_{k,\delta}(u) = \phi_\delta \int_0^u m_{k,\delta}(u - y) f_\delta(y)dy + v_{k,\delta}(u), \quad (2.12)$$

*where* $\phi_\delta = \int_0^\infty \int_0^\infty h_\delta(x, y|0)dxdy$,

$$f_\delta(y) = \frac{1}{\phi_\delta} \int_0^\infty h_\delta(x, y|0)dx$$

*and*

$$\begin{aligned} v_{k,\delta}(u) = &\sum_{j=1}^k \binom{k}{j} \int_0^u m_{k-j,\delta}(u - y) \int_0^\infty h^{*j}_\delta(x, y|0)dxdy \\ &+ \int_u^\infty \int_0^\infty w(x + u, y - u, u) h^{*k}_{1,\delta}(x, y|0)dxdy \\ &+ \int_u^\infty \int_0^\infty \int_0^x w(x + u, y - u, v + u) \\ &\times h^{*k}_{2,\delta}(x, y, v|0)dvdxdy. \end{aligned} \quad (2.13)$$

*For* $k = 0$, *(2.12) reduces to (2.6).*

**Proof.** First, rewrite (2.6) as

$$m_\delta(u) = \int_0^u m_\delta(u - y) f^{**}_\delta(y)dy + v_\delta(u), \quad (2.14)$$

where $f^{**}_\delta(y) = \phi_\delta f_\delta(y) = \int_0^\infty h_\delta(x, y|0)dx$.

Download English Version:

https://daneshyari.com/en/article/5076589

Download Persian Version:

https://daneshyari.com/article/5076589

Daneshyari.com