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Kernel-type estimator of the reinsurance premium for heavy-tailed loss distributions

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HIGHLIGHTS

- We generalize the classical estimator of the reinsurance premium for heavy-tailed losses.
- We propose a bias-reduced estimator of the reinsurance premium.
- The asymptotic normality of the given estimators is established.
- A small simulation study illustrates the performance of our approach.

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ABSTRACT

In this paper, we generalize the classical estimator of the reinsurance premium for heavy-tailed loss distributions with a kernel-type estimator. Since this estimator exhibits a bias, we propose its bias-reduced version by using a least-squares method. The asymptotic normality of the proposed estimators is established under suitable assumptions. A small simulation study is carried out to prove the performance of our approach.

1. Introduction

The major worry for the insurance and reinsurance companies is to determine the adequate premium. In the insurance literature, there exist several premium principles such as: expected value, variance and value-at-risk. For more details on premium principles and their properties, we refer to Goovaerts et al. (1984). Wang (1996) proposed a premium principle named proportional hazard premium (PHP) of an insured risk X, a non-negative random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with continuous distribution function F, depends on the hazard function $S = 1-F$ and a parameter $r \geq 1$ called the risk aversion index or the distortion parameter. The PHP is defined as follows

$$\Pi_r = \int_0^\infty (S(x))^{1/r}\, dx.$$

In some actuarial problems, as in the reinsurance treaty, one is interested in the estimation of a premium for a given retention level $R > 0$ notation $\Pi_{r,R}$, that is, a reinsurance premium of the high layer $[R, \infty)$. This type of problem can be found whenever the insured represents a dangerous level of risk for the insurance company, and decides to give a part of this loss to another reinsurance company, because it may not have sufficient capital to cover the total risk. The reinsurance premium of the high layer is defined as follows

$$\Pi_{r,R} = \int_R^\infty (S(x))^{1/r}\, dx.$$

For heavy-tailed distributions, Beirlant et al. (2001), Necir and Boukhetala (2004), Vandewalle and Beirlant (2006) and Necir et al. (2007) have introduced and studied different estimators for $\Pi_{r,R}$, in the case of high-excess loss layers ($R \to \infty$).

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A distribution function F is said to be heavy-tailed whenever the tail function $1-F$ is a regularly varying function with index $(-1/\gamma)<0$, i.e., for any $x>0$,

$$1-F(x)=x^{-1/\gamma}\mathbb{L}(x),$$

where $\mathbb{L}$ is a slowly varying function at infinity, that is, $\mathbb{L}(tx)/\mathbb{L}(t)\to 1$ as $t\to\infty$. The class of regularly varying functions includes popular distributions such those Pareto's, Burr's, Student's, Fréchet's, α-stable $(0<\alpha<2)$, and log-gamma, which are known to be appropriate models of fitting large insurance claims, large fluctuations of prices, log-returns, and so on (see Beirlant et al., 2001).

Let $X_{1,n}\le\cdots\le X_{n,n}$, $n\ge 1$, be the order statistics pertaining to a sample $X_1,\ldots,X_n$ from X and let $k=k_n$ be an integer sequence satisfying

$$1<k<n, k\to\infty, \quad \text{and} \quad k/n\to 0 \quad \text{as } n\to\infty.$$

Let, for $0<s<1$, $Q(s)=\inf\{x:F(x)\ge s\}$ be the quantile function pertaining to F. At an optimal retention level $R=R_{opt}=Q(1-k/n)$, the semi-parametric estimator for $\Pi_{r,R}$ that proposed by Necir et al. (2007) is

$$\widetilde{\Pi}_{r,\widehat{R}_{opt}}=(k/n)^{1/r}\frac{r}{1/\widehat{\gamma}^{H}_{n,k}-r}X_{n-k,n}, \quad \text{for } \widehat{\gamma}^{H}_{n,k}<1/r,$$

where $\widehat{R}_{opt}=X_{n-k,n}$ and $\widehat{\gamma}^{H}_{n,k}$ is the classical Hill estimator (Hill, 1975) of the tail index γ, defined by

$$\widehat{\gamma}^{H}_{n,k}=\frac{1}{k}\sum_{i=1}^{k}i\left(\log X_{n-i+1,n}-\log X_{n-i,n}\right).$$

A major drawback of the Hill estimator is the discrete character of its behavior in the sense that increasing k by 1, can change the actual value of the estimate considerably. Using a kernel function K, Csörgő et al. (1985) proposed a smoother version of Hill's estimator defined by

$$\widehat{\gamma}^{K}_{n,k}=\frac{1}{k}\sum_{i=1}^{k}K\left(\frac{i}{k+1}\right)Z_i,$$

where $Z_i=i\left(\log X_{n-i+1,n}-\log X_{n-i,n}\right)$. The class of kernel estimators $\widehat{\gamma}^{K}_{n,k}$ generalizes the Hill estimator. Note that, using the uniform kernel $K=\underline{K}=\mathbf{1}_{(0,1)}$ yields Hill's estimator $\widehat{\gamma}^{H}_{n,k}$ as a special case.

In this paper, we propose a kernel-type estimator for the reinsurance premium $\Pi_{r,R_{opt}}$ of a heavy-tailed distribution. Thus, $\Pi_{r,R_{opt}}$ can be estimated by

$$\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}=(k/n)^{1/r}\frac{r}{1/\widehat{\gamma}^{K}_{n,k}-r}X_{n-k,n}, \quad \text{for } \widehat{\gamma}^{K}_{n,k}<1/r. \tag{1.1}$$

The rest of this paper is organized as follows. In Section 2, we study the asymptotic properties of $\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$ and propose its bias-reduced version whose asymptotic normality is also obtained. In Section 3, we perform a small simulation study, by sampling from Fréchet distribution, to compare these estimators. All proofs are given in Section 4.

2. Main results

Firstly, in this section, we study the asymptotic normality of $\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$.

2.1. Asymptotic normality of $\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$

From (1.1), it is clear that the asymptotic normality of $\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$ is related to $\widehat{\gamma}^{K}_{n,k}$. To establish such a type of result, as usual in the extreme value theory, we need a second-order condition on the tail quantile function $\mathbb{U}$ defined, for $1<t<\infty$, as

$$\mathbb{U}(t)=(1/(1-F))^{-1}(t)=Q(1-1/t).$$

We say that the function $\mathbb{U}$ fulfills the second-order regular variation condition with second-order parameter $\rho<0$ if there exists a function $A(t)$ tending to 0 and not changing sign near infinity, such that for all $x>0$

$$\lim_{t\to\infty}\frac{\log\mathbb{U}(tx)-\log\mathbb{U}(t)-\gamma\log x}{A(t)}=\frac{x^{\rho}-1}{\rho}. \tag{2.2}$$

We also need the following classical conditions about the kernel K.

Condition $(\mathbb{K})$. Let K be a function defined on $(0,1]$.

(i) $K(s)\ge 0$, whenever, $0<s\le 1$ and $K(1)=K'(1)=0$.
(ii) $K(\cdot)$ is differentiable, non-increasing and right continuous on $(0,1]$.
(iii) K and K' are bounded.
(iv) $\int_0^1 K(u)du=1$.
(v) $\int_0^1 u^{-1/2}K(u)du<\infty$.

Theorem 2.1. *Let F be a distribution function satisfying (2.2) with $\gamma\in(1/2,1)$ and suppose that $(\mathbb{K})$ holds. Let $k=k_n$ be an integer sequence satisfying $k\to\infty$, $k/n\to 0$ and $\sqrt{k}A(n/k)=O(1)$ as $n\to\infty$. For any $1\le r<1/\gamma$, on the probability space $(\Omega,\mathcal{A},\mathbb{P})$, there exists a sequence of Brownian bridges $\{\mathbb{B}_n(s);0\le s\le 1\}$ such that, as $n\to\infty$,*

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}-\Pi_{r,R_{opt}}\right)=\sqrt{k}A(n/k)\mathcal{AB}_K(\gamma,r,\rho)+\mathcal{W}_{1,n}+\mathcal{W}_{2,n}(K)+o_{\mathbb{P}}(1),$$

where

$$\mathcal{AB}_K(\gamma,r,\rho)=\frac{r}{1-r\gamma}\left(\frac{1}{r\gamma+r\rho-1}+\frac{1}{1-r\gamma}\int_0^1 s^{-\rho}K(s)\,ds\right),$$

and

$$\begin{cases}\mathcal{W}_{1,n}=-\dfrac{r\gamma^2}{1-r\gamma}\sqrt{\dfrac{n}{k}}\mathbb{B}_n\left(1-\dfrac{k}{n}\right),\\ \mathcal{W}_{2,n}(K)=\dfrac{r\gamma}{(1-r\gamma)^2}\sqrt{\dfrac{n}{k}}\displaystyle\int_0^1 s^{-1}\mathbb{B}_n\left(1-\frac{sk}{n}\right)d(sK(s)).\end{cases}$$

Corollary 2.1. *Under the assumptions of Theorem 2.1, if $\sqrt{k}A(n/k)\to\lambda\in\mathbb{R}$, we have*

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}-\Pi_{r,R_{opt}}\right)\xrightarrow{\mathcal{D}}\mathcal{N}\left(\lambda\mathcal{AB}_K(\gamma,r,\rho),\mathcal{AV}_K(\gamma,r)\right), \quad \text{as } n\to\infty,$$

where

$$\mathcal{AV}_K(\gamma,r)=\frac{r^2\gamma^4}{(1-r\gamma)^2}+\frac{r^2\gamma^2}{(1-r\gamma)^4}\int_0^1 K^2(s)ds.$$