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Efficient approximations for numbers of survivors in the Lee–Carter model

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ABSTRACT

In portfolios of life annuity contracts, the payments made by an annuity provider (an insurance company or a pension fund) are driven by the random number of survivors. This paper aims to provide accurate approximations for the present value of the payments made by the annuity provider. These approximations account not only for systematic longevity risk but also for the diversifiable fluctuations around the unknown life table. They provide the practitioner with a useful tool avoiding the problem of simulations within simulations in, for instance, Solvency 2 calculations, valid whatever the size of the portfolio.

1. Introduction and motivation

In this paper, we consider the present value of life annuity payments accounting for the stochastic nature of decrements. Precisely, the systematic longevity risk coming from the unknown underlying life table as well as the theoretically diversifiable risk of random fluctuations around this life table are both taken into account. Thus, the size of the portfolio enters the calculations and this dimension is very important for small to medium-sized portfolios (see, e.g., Donnelly, 2011).

In the literature, the case of life annuity policies has been treated quite extensively but only in the limiting case, for homogeneous portfolios comprising infinitely many (conditionally) independent contracts. The applicability of these limiting results may be questioned in insurance practice as life annuity portfolios do not always contain enough policies to reach full diversification. For these reasons, Hoedemakers et al. (2005) proposed to approximate the distribution of the number of survivors using the Normal Power formula. In this paper, we pursue this idea and we allow for unknown future mortality improvements, the death probabilities prevailing in the future being difficult to assess.

After Lee and Carter (1992), we assume that the death rate at age x in calendar year t is of the form $\exp(\alpha_x + \beta_x\kappa_t)$. Here, the time index κ_t reflects the general level of mortality and the age-specific component β_x represents how rapidly or slowly mortality at each age varies when the general level of mortality changes. The dynamics of the time index is usually described by ARIMA models. Conditional survival probabilities, given the time index future trajectory, are complicated functions of the κ_ts. As there is no analytical expression available for their distribution function, Denuit and Dhaene (2007) used comonotonicity to approximate the distribution of the sums of strongly correlated LogNormal random variables playing a central role in the Lee–Carter framework. Expanding on this approach, Denuit (2008) derived analytic approximations for the quantiles of the life annuity conditional expected present value given the κ_ts. This is made by supplementing the comonotonic approximations for the conditional survival probabilities worked out in Denuit and Dhaene (2007) with a second approximation of the same type for the life annuity conditional expected present value, given the κ_ts. Denuit et al. (2010) further studied the quality of these approximations, allowing for general ARIMA models instead of the simple random walk with drift adopted in the majority of papers using Lee–Carter methodology.

In this paper, our aim is to develop accurate approximations for the present value of the payments made in favor of a group of n annuitants. The size n of the group explicitly enters the computations so that our results apply also to small portfolios. Deriving the exact distribution for the present value of life annuity payments requires

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extensive simulations or numerical evaluations. The approximations derived in this paper after Denuit and Dhaene (2007) and Denuit (2008) avoid the requirement to conduct simulations within simulations in, for instance, Solvency 2 reserving calculations. Numerical illustrations show that the comonotonic approximations perform well, which suggests that they can be used in practice to evaluate the consequences of the uncertainty in future death rates.

To derive an effective comonotonic approximation, it is essential to identify in the problem under consideration random variables that are as much positively correlated as possible. Partial sums are often good candidates in that respect, as demonstrated in Denuit and Dhaene (2007). For portfolios of life annuities, the numbers of survivors up to times $1, 2, 3, \ldots$ form a strongly positively dependent sequence for which the comonotonic approximation is expected to work well. This is precisely the intuitive idea exploited in the present paper, which turns out to provide accurate approximations. Notice that making the lifetimes comonotonic in a homogeneous portfolio means that all policyholders die at the same time, which is very crude. Hence, it is important to select appropriately the random variables which will be replaced by their comonotonic versions.

The paper is organized as follows. In Section 2, we briefly recall the comonotonic approximations for the conditional survival probabilities derived by Denuit and Dhaene (2007) and for the conditional expectation of annuity payments present value derived by Denuit (2008). We supplement previous results with increasing directionally convex stochastic inequalities between the Lee–Carter conditional survival probabilities and their approximations. Section 3 proposes new approximations for the consecutive numbers of survivors. It is established there that the approximate numbers of survivors dominate the Lee–Carter ones in the increasing directionally convex order, which allows us to derive stop-loss order stochastic inequalities for the present value of life annuity payments. Numerical illustrations are discussed in Section 4. Section 5 briefly concludes.

2. Comonotonic approximations

2.1. Conditional survival probabilities

In this paper, we assume that the force of mortality at age x and time t, denoted as $\mu_x(t)$, is constant within bands of age and time in the Lexis diagram, but allowed to vary from one band to the next. Specifically, given any integer age x and calendar year t, it is supposed that

$$\mu_{x+\xi}(t+\tau) = \mu_x(t) \quad \text{for } 0 \le \xi, \tau < 1. \tag{2.1}$$

Furthermore, the force of mortality is of the form

$$\ln \mu_x(t) = \alpha_x + \beta_x \kappa_t. \tag{2.2}$$

Henceforth, we assume that the values $\kappa_1, \ldots, \kappa_{t_0-1}$ are known but that $\kappa_{t_0}, \kappa_{t_0+1}, \ldots$ are unknown and have to be projected from some appropriate time series model. The future trajectory $\kappa_{t_0}, \kappa_{t_0+1}, \ldots$ is henceforth denoted as $\boldsymbol{\kappa}$. Therefore, the force of mortality $\mu_x(t)$ given in (2.2) is not constant but develops over time following a stochastic process. Also, we assume that $\beta_x \ge 0$ for every age x so that κ_t has an unambiguous effect on mortality. The positivity of the β_xs is supported by empirical evidence.

Consider an individual aged x_0 in calendar year t_0, with remaining lifetime T subject to (2.1)–(2.2). Define $\delta_j = \exp(\alpha_{x_0+j}) > 0$, $Z_j = \beta_{x_0+j}\kappa_{t_0+j}$ and

$$S_d = \sum_{j=0}^{d-1} \exp\Big(\alpha_{x_0+j} + \beta_{x_0+j}\kappa_{t_0+j}\Big) = \sum_{j=0}^{d-1} \delta_j \exp(Z_j).$$

In the applications, the time index is generally modeled by means of ARIMA time series models. Hence, we assume that $\boldsymbol{\kappa}$ is multivariate Normal so that we have $Z_j \sim \mathcal{N}or(\mu_j, \sigma_j^2)$. Then, the conditional survival probability over the next d years, given the future trajectory $\boldsymbol{\kappa}$ of the time index is given by

$$\Pr[T > d|\boldsymbol{\kappa}] = \exp(-S_d) = {}_dP_{x_0}(t_0), \quad d = 1, 2, \ldots.$$

Denuit and Dhaene (2007) proposed comonotonic approximations for the conditional survival probabilities ${}_dP_{x_0}(t_0)$. Specifically, these conditional probabilities are expected to be closely dependent for increasing values of d since they can be viewed as the exponential of the sum of death rates from age x_0 to age $x_0 + d - 1$. So, it may be reasonable to approximate the random vector of conditional survival probabilities with its comonotonic version.

Recall that a random vector $(X_1, \ldots, X_d)$ is said to be comonotonic if, and only if, there exist a random variable Z and non-decreasing functions $g_1, \ldots, g_d$, such that $(X_1, \ldots, X_d)$ is distributed as $\big(g_1(Z), \ldots, g_d(Z)\big)$. Equivalently, $(X_1, \ldots, X_d)$ is comonotonic if it is distributed as $\big(g_1(Z), \ldots, g_d(Z)\big)$ with $g_1, \ldots, g_d$ non-increasing. In particular, we may choose Z to be uniformly distributed over the unit interval $[0, 1]$ and g_i to be the quantile function of X_i, i.e. the left-continuous inverse of the distribution function of X_i. A detailed account of comonotonicity can be found in Dhaene et al. (2002a,b) and Denuit et al. (2005).

In order to determine whether the approximations derived in this paper are conservative, we can use the following stochastic order relations. For more details, the readers are referred, e.g., to Denuit et al. (2005). Considering two random variables X and Y, X is said to be smaller than Y in the increasing convex order, or stop-loss order, henceforth denoted as $X \preceq_{\text{icx}} Y$, if the inequality $\mathbb{E}[g(X)] \le \mathbb{E}[g(Y)]$ holds true for all the non-decreasing and convex functions g for which the expectations exist. A usual strengthening of the stop-loss order is obtained by requiring in addition that the means of the random variables to be compared are equal. More precisely, X is said to be smaller than Y in the convex order, henceforth denoted by $X \preceq_{\text{cx}} Y$, if $\mathbb{E}[X] = \mathbb{E}[Y]$ and $X \preceq_{\text{icx}} Y$ simultaneously hold. The term "convex" is used since $X \preceq_{\text{cx}} Y \Leftrightarrow \mathbb{E}[g(X)] \le \mathbb{E}[g(Y)]$ for all the convex functions g for which the expectations exist.

Stochastic orderings $\preceq_{\text{cx}}$ and $\preceq_{\text{icx}}$ aim to mathematically express the intuitive ideas of "being less variable than" and "being smaller and less variable than" for random variables. Dealing with random vectors, $\preceq_{\text{cx}}$ and $\preceq_{\text{icx}}$ may apply marginally to each component but we also need multivariate stochastic order relations that translate the fact that the components of one of these vectors are "more positively dependent" than those of the other random vector. The supermodular order translates this idea in mathematical terms. Precisely, recall that a function $g : \mathbb{R}^d \to \mathbb{R}$ is said to be supermodular if the inequality

$$\begin{aligned} g(&x_1, \ldots, x_i + \epsilon, \ldots, x_j + \delta, \ldots, x_d) \\ &- g(x_1, \ldots, x_i + \epsilon, \ldots, x_j, \ldots, x_d) \\ \ge g(&x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_d) \\ &- g(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_d) \end{aligned}$$

holds for all $\boldsymbol{x} \in \mathbb{R}^d$, $1 \le i < j \le d$ and all $\epsilon, \delta > 0$. If the function is regular enough then supermodularity corresponds to $\frac{\partial^2}{\partial x_i \partial x_j} g \ge 0$ for every $i \ne j$. Now, consider two d-dimensional random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $\mathbb{E}[g(\boldsymbol{X})] \le \mathbb{E}[g(\boldsymbol{Y})]$ for all supermodular functions $g : \mathbb{R}^d \to \mathbb{R}$, provided the expectations exist. Then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the supermodular order, which is denoted by $\boldsymbol{X} \preceq_{\text{sm}} \boldsymbol{Y}$. In words, $\boldsymbol{X} \preceq_{\text{sm}} \boldsymbol{Y}$ means that $X_1, \ldots, X_d$ are less positively related than $Y_1, \ldots, Y_d$. Notice that $\boldsymbol{X} \preceq_{\text{sm}} \boldsymbol{Y} \Rightarrow X_i$ and Y_i are identically distributed for each i so that $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same