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Optimal investment, consumption and proportional reinsurance for an insurer with option type payoff[☆]

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HIGHLIGHTS

- An investment, consumption and proportional reinsurance problem with option type payoff is considered.
- Our problem is discussed in a general diffusion framework.
- Very general constraints are imposed on the investment and reinsurance rate.
- The problem is solved by using backward stochastic differential equations.
- Some special cases are studied in detail by using Malliavin calculus.

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ABSTRACT

This paper is devoted to the study of optimization of investment, consumption and proportional reinsurance for an insurer with option type payoff at the terminal time under the criterion of exponential utility maximization. The surplus process of the insurer and the financial risky asset process are assumed to be diffusion processes driven by Brownian motions which are non-Markovian in general. Very general constraints are imposed on the investment and the proportional reinsurance processes. Based on the martingale optimization principle, we use BSDE and BMO martingale techniques to derive the optimal strategy and the optimal value function. Some interesting particular cases are studied in which the explicit expressions for the optimal strategy are given by using the Malliavin calculus.

1. Introduction

In order to control and manage risk, reinsurance and investment are two effective ways for insurers. As a result, optimal reinsurance and investment problems for insurers have attracted much attention in the actuarial literature in recent years. To study the optimal investment–reinsurance policies for an insurer, maximization of exponential utility, among many other criteria, is widely adopted by researchers. See Bai and Guo (2008), Cao and Wan (2009), Zhang et al. (2009), Liang et al. (2011), Liang and Bayraktar (2014) and the references therein.

In the papers mentioned above, the main approaches to study the optimization problems are stochastic control theory and related methodologies, especially the HJB equation method. That is a quite effective method for many stochastic optimization problems in insurance. However, the Bellman programming principle, which leads to the HJB equations, can only be applied to Markovian state processes. This implies that the HJB equation method cannot be used directly when the state process is assumed to be a non-Markovian process which is the case we are interested in.

In the present paper, we consider an exponential utility maximization problem for an insurer with a consumption process, who invests in financial markets and takes the proportional reinsurance business. Meanwhile, we also consider the case where the wealth process is of a terminal option type payoff, since option type payoff can be used to reduce risk. We will allow the parameter processes to be general stochastic processes. Therefore, the wealth process of

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the insurer with investment, consumption and proportional reinsurance is non-Markovian in general. Instead of adopting the HJB equation method, we use BSDE and BMO martingale techniques to solve our problem. Under very general constraints on the proportional reinsurance rate and the investment, we can derive the optimal strategy in a closed form. These constraints are weaker than those employed by the papers mentioned above. In particular, when there is no constraint on the investment process and the reinsurance rate is assumed to be nonnegative, or no reinsurance is considered, we can obtain the explicit expressions for the optimal strategies by using the Malliavin calculus.

As a matter of fact, the BSDE and BMO martingale techniques were used to handle the utility maximization problem with option type payoff in finance. Hu et al. (2005) first studied the utility maximization problem in an incomplete financial market by using BSDE and BMO martingale techniques. Later, Morlais (2009a,b) extended the risk asset process driven by the Brownian motion to that driven by general continuous martingales and the Lévy process respectively. By using the BSDE with jumps, Lim and Quenez (2011) studied the exponential utility maximization problem in an incomplete financial market with defaults. Cheridito and Hu (2011) studied an optimal consumption and investment problem under utility maximization with general stochastic constraints by using BSDE and BMO martingale techniques. However, so far, we have not found any report in the literature on the study of exponential utility maximization problem for an insurer by using BSDE and BMO martingale techniques.

It should be mentioned that there exist some works about the optimal investment and proportional reinsurance problem under the criterion of utility maximization in the general non-Markovian framework where the parameter processes are supposed to be very general stochastic processes. For example, by a modified "duality method", Liu and Ma (2009) studied a utility optimization problem for a general proportional reinsurance and investment model, where the reverse process of the insurer is non-Markovian in general. Peng and Hu (2013) investigated the optimal investment and proportional reinsurance under partial information with the criterion of utility maximization in a non-Markovian framework by using the Malliavin calculus. However, the methods used in these two papers are not applicable to the problem in the present paper, since the investment and proportional reinsurance rate in our model are subject to very general constraints, not necessarily convex.

The rest of the paper is organized as follows. Section 2 gives the model dynamics and states the problem. Section 3 is devoted to the solution to the problem by using BSDE and BMO martingale techniques. Section 4 studies the optimal strategies for some particular cases by using the Malliavin calculus. Finally, Section 5 concludes this paper.

2. The dynamics and the optimization problem

We consider a continuous time model where all uncertainties come from a complete probability space $(\Omega, \mathcal{F}, P)$. Here P is a real world probability. Fix $T > 0$ as the terminal time. Suppose that the surplus process before investment of an insurer satisfies

$$\begin{cases} dl_t = p(t)dt + q(t)dW_t^1, \\ l_0 = x. \end{cases}$$

Here, the constant $x > 0$ is the initial surplus, W^1 is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$. Denote $\mathcal{F}_t^1 = \sigma(W_s^1, 0 \leqslant s \leqslant t) \vee \mathcal{N}$, where $\mathcal{N}$ is the P null sets. Assume that p and q are bounded $(\mathcal{F}_t^1)_{0\leqslant t\leqslant T}$ adapted cadlag processes.

The insurer can invest in the financial market with no transaction cost and tax where two investment possibilities are available:

- a risk free asset $A(t)$ with price dynamics:

$$\begin{cases} dA(t) = rA(t)dt, \\ A_0 = 1; \end{cases}$$

- a risky asset $S(t)$ with price dynamics:

$$\begin{cases} dS(t) = S(t)[\mu(t)dt + \sigma(t)dW_t^2], \\ S(0) > 0, \end{cases}$$

where W^2 is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$ independent of W^1. Denote $\mathcal{F}_t^2 = \sigma(W_s^2, 0 \leqslant s \leqslant t) \vee \mathcal{N}$. Suppose that the interest rate $r > 0$ is a constant, while the return rate $\mu(\cdot)$ and the volatility $\sigma(\cdot)$ are bounded $(\mathcal{F}_t^2)_{0\leqslant t\leqslant T}$ adapted cadlag processes. Let b_t denote the total amount of money invested in the risky asset at time t, and $R_t - b(t)$ is invested in the risk free asset. We also consider the consumption of the insurer. Let c_t be the consumption rate at time t. Besides investment and consumption, the insurer is allowed to purchase proportional reinsurance or acquire new business (for example, acting as a reinsurer of other insurers, see Bäuerle, 2005) at each moment in order to reduce insurance business risk. The proportional reinsurance/new business level is associated with the value of risk exposure $a(t) \in [0, +\infty)$ at any time $t \in [0, T]$. $a(t) \in [0, 1]$ corresponds to a proportional reinsurance cover and implies that the insurer should pay $a(t)Y$ for the claim Y occurring at time t while the reinsurer should pay the rest $(1 - a(t))Y$. $a(t) > 1$ means that the insurer can take an extra insurance business from other companies (i.e., act as a reinsurer for other cedents). For convenience, we call the process of risk exposure $a(t),\ t \in [0, T]$, as a reinsurance policy. Under the above assumptions, the wealth process with investment, consumption and reinsurance evolves over time as follows:

$$dR_t = \Big[p(t) - \big(1 - a(t)\big)\lambda(t) + r(R_t - b(t)) + \mu(t)b(t) - c(t)\Big]dt + a(t)q(t)dW_t^1 + b(t)\sigma(t)dW_t^2 \quad (2.1)$$

where $\lambda(t)$ represents the safety loading premium of reinsurance satisfying $\lambda(t) \geqslant p(t) \geqslant 0$ a.s., for all $t \in [0, T]$. Assume that $\sigma(t),\ q(t) > 0$ a.s., for all $t \in [0, T]$. Let $\theta_1(t) = \frac{\lambda(t)}{q(t)}$ and $\theta_2(t) = \frac{\mu(t)-r}{\sigma(t)}$. θ_1 and θ_2 are assumed to be bounded processes. Denote $u_1(t) = q(t)a(t),\ u_2(t) = \sigma(t)b(t),\ \pi(t) = \big(u_1(t), u_2(t), c(t)\big),\ 0 \leqslant t \leqslant T$. In what follows, we consider π as the optimal strategy. Then (2.1) can be rewritten as

$$dR_t^\pi = [p(t) - \lambda(t) - c(t)]dt + rR_t^\pi dt + u_1(t)[dW_t^1 + \theta_1(t)dt] + u_2(t)[dW_t^2 + \theta_2(t)dt]. \quad (2.2)$$

If

$$\int_0^T \big[|p(t) - \lambda(t) - c(t)| + u_1^2(t) + u_2^2(t)\big]dt < \infty, \quad P\text{-a.s.},$$

then the stochastic differential equation (2.2) has a unique continuous solution R_t^π.

The insurer must provide at maturity T an option type payoff represented by a bounded $\mathcal{F}_T^2$-measurable random variable ξ. Suppose that its risk aversion is characterized by an exponential utility:

$$U(x) = -\exp(-\upsilon x),$$

where $\upsilon > 0$ represents the risk aversion rate. This utility function plays a crucial role in insurance context, since it is the only function under which the principle of zero utility gives a fair premium that is independent of the level of reserve of an insurance company (see Gerber, 1979). Let $\bar{K}_1,\ \bar{K}_2$ be closed (not necessarily convex) subsets of $\mathbb{R}^+$ and $\mathbb{R}$ respectively. The distance between the point $a \in \mathbb{R}$ to the set $\bar{K}_i$ is defined by

$$\operatorname{dist}(a, \bar{K}_i) = \min_{x\in\bar{K}_i} |a - x|, \quad i = 1, 2.$$