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Optimal reinsurance with regulatory initial capital and default risk

Jun Cai, Christiane Lemieux, Fangda Liu *

Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

HIGHLIGHTS

- Proposing a reinsurance model with regulatory initial reserve and default risk.
- Deriving optimal reinsurance strategies in the proposed model for insurers.
- The regulatory reserve and default risk have a significant impact on the strategies.
- The strategies are more complicated than those in the default risk-free models.

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ABSTRACT

In a reinsurance contract, a reinsurer promises to pay the part of the loss faced by an insurer in exchange for receiving a reinsurance premium from the insurer. However, the reinsurer may fail to pay the promised amount when the promised amount exceeds the reinsurer's solvency. As a seller of a reinsurance contract, the initial capital or reserve of a reinsurer should meet some regulatory requirements. We assume that the initial capital or reserve of a reinsurer is regulated by the value-at-risk (VaR) of its promised indemnity. When the promised indemnity exceeds the total of the reinsurer's initial capital and the reinsurance premium, the reinsurer may fail to pay the promised amount or default may occur. In the presence of the regulatory initial capital and the counterparty default risk, we investigate optimal reinsurance designs from an insurer's point of view and derive optimal reinsurance strategies that maximize the expected utility of an insurer's terminal wealth or minimize the VaR of an insurer's total retained risk. It turns out that optimal reinsurance strategies in the presence of the regulatory initial capital and the counterparty default risk are different both from optimal reinsurance strategies in the absence of the counterparty default risk and from optimal reinsurance strategies in the presence of the counterparty default risk but without the regulatory initial capital.

1. Introduction

Reinsurance is an important risk management tool for an insurer and has been an interesting research topic in actuarial science. In a static reinsurance model or one-period reinsurance model, one assumes that the underlying (aggregate) loss faced by an insurer in a fixed time period is a non-negative random variable X with survival function $S_X(x) = \Pr\{X > x\} = 1 - F_X(x)$. In a reinsurance contract, a reinsurer agrees to pay the part of the loss X, denoted by $I(X)$, to the insurer at the end of the contract term, while the insurer will pay a reinsurance premium, denoted by P_I, to the reinsurer when the contract is signed, where the function $I(x)$ is called ceded loss function or indemnification function. Thus, under the reinsurance contract I, the retained loss for the insurer is $R(X) = X - I(X)$, where the function $R(x) = x - I(x)$ is called retained loss function. In order to avoid any moral issue, a feasible reinsurance contract I should satisfy the following two conditions:

(1) $I : [0, \infty) \to [0, \infty)$ such that $I(0) = 0$ and I is non-decreasing;
(2) $I(y) - I(x) \le y - x$, for any $0 \le x \le y$.

These two conditions imply that both $I(x)$ and $R(x)$ are continuous and non-decreasing on $[0, \infty)$. The first condition means that the larger is the incurred loss by an insurer, the larger is the covered loss by a reinsurer. The second condition implies that the growth rate of the covered loss by a reinsurer should not be faster than the growth rate of the underlying loss faced by an insurer.

Throughout this paper, we denote the set of all feasible reinsurance contracts satisfying conditions (1) and (2) by $\mathcal{I}$ and define $(a)^+ = \max\{a, 0\}$, $a \wedge b = \min\{a, b\}$, and $a \vee b = \max\{a, b\}$. In addition, we interpret the term "increasing" to mean "non-decreasing", while "decreasing" means "non-increasing".

* Corresponding author. Tel.: +1 5195883816.
E-mail address: f27liu@uwaterloo.ca (F. Liu).

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The purpose of optimal reinsurance design is to find ceded loss functions I^*, which are optimal under certain optimization criteria. Optimal reinsurance from an insurer's point of view has been studied extensively in the literature. Two commonly used optimization criteria are maximizing the expected utility of an insurer's terminal wealth and minimizing the risk measure of an insurer's total retained risk. Some recent references on optimal reinsurance under different risk measures include Balbás et al. (2009), Asimit et al. (2013a,b), Cai et al. (2008), Cheung (2010), Chi (2012), Chi and Tan (2011), and so on. In addition, the one-period reinsurance model with one loss variable has been extended to models with more insurance lines of business or to models that discuss the interests of both insurers and reinsurers. Recent references on these issues can be found in Cai and Wei (2012), Cai et al. (2013), Cheung et al. (2014a), Hürlimann (2011), and references therein.

In most studies on optimal reinsurance, one assumes that a reinsurer will pay the promised loss $I(X)$ regardless of its solvency or equivalently, one ignores the potential default by a reinsurer. Indeed, default risk can be reduced if a reinsurer has a sufficiently large initial capital or reserve. However, default might occur even if the initial capital of a reinsurer is very large. In a reinsurance contract I, a reinsurer may fail to pay the promised amount $I(X)$ or a reinsurer may default due to different reasons. One of the main reasons could be that the promised amount $I(X)$ exceeds the reinsurer's solvency. The larger is the initial reserve of a reinsurer, the smaller is the likelihood that default will occur. This is why the initial capital of a seller (reinsurer) of a reinsurance contract should meet some requirements by regulations to reduce default risk.

Recently, counterparty default risks in reinsurance designs or other related studies have been discussed in Asimit et al. (2013a,b), Bernard and Ludkovski (2012), Burren (2013), Cummins et al. (2002), Dana and Scarsini (2007), Menegatti (2009), and references therein. Several models with default risks have been proposed in these references. However, in the references for reinsurance designs with default risks such as Asimit et al. (2013a,b), Bernard and Ludkovski (2012), and so on, they assume a constant initial capital or reserve for a reinsurer regardless of how large a reinsurer's promised amount $I(X)$ is, or they do not consider the influence of a reinsurer's initial reserve on optimal reinsurance strategies. Indeed, a reasonable requirement on a reinsurer could be that the larger is the promised indemnity of a reinsurer, the larger the initial reserve of a reinsurer should be.

In this paper, we propose a reinsurance model with regulatory initial capital and default risk. We assume that the initial capital or reserve of a seller (reinsurer) of a reinsurance contract I is determined through regulation by the value-at-risk (VaR) of its promised indemnity $I(X)$, and denote the initial capital of the reinsurer by $\omega_I = \mathrm{VaR}_\alpha(I(X))$, where $\mathrm{VaR}_\alpha(Z) = \inf\{z : \Pr\{Z > z\} \le \alpha\}$ is the VaR of a random variable Z and $0 < \alpha < 1$ is called the risk level. Usually, α is a small value such as $\alpha = 0.01$ or 0.05. We assume that the reinsurer charges a reinsurance premium P_I based on the promised indemnity $I(X)$. The insurer is aware of the potential default by the reinsurer but the worst case for the insurer is that the reinsurer only pays $\omega_I + P_I$ if $I(X) > \omega_I + P_I$. Thus, when the insurer is seeking for optimal reinsurance strategies and taking account of the potential default by the reinsurer, the insurer assumes the worst indemnity $I(X) \wedge (\omega_I + P_I)$ from the reinsurer. Indeed, when $\omega_I = \mathrm{VaR}_\alpha(I(X))$, we know $\Pr\{I(X) > \omega_I + P_I\} \le \alpha$ or the probability of default by the reinsurer is not greater than the value α, which could be an acceptable risk level for the insurer. Hence, under the proposed reinsurance model, the total retained risk or cost of the insurer is $X - I(X) \wedge (\omega_I + P_I) + P_I$ and the insurer's terminal wealth is $w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I$, where w_0 is the initial capital of the insurer.

We point out that in the above proposed model, the minimum or guaranteed available capital of the reinsurer at the end of the contract is the (regulatory) initial reserve plus the reinsurance premium. However, the actual available capital of the reinsurer at the end of the contract may be different from the initial reserve plus the reinsurance premium. For example, the actual available capital of the reinsurer may be higher than the initial reserve plus the reinsurance premium if the reinsurer can use the capitals or reserves from its other portfolios or if the reinsurer has investment profits on the initial reserve and/or the reinsurance premium or if the reinsurer has other assets. On the other hand, the actual available capital of the reinsurer may be lower than the initial reserve plus the reinsurance premium if the reinsurer spends some of the initial reserve and/or the reinsurance premium or if the reinsurer has investment losses on the initial reserve and/or the reinsurance premium. Each of these scenarios may result in different reinsurance models. Indeed, our proposed model is just one of many possible mathematical models for reinsurance designs. In our proposed model, we emphasize that the initial reserve of the reinsurer is determined by the VaR of the reinsurer's promised indemnity due to regulatory requirements, the insurer believes that the guaranteed or minimum available capital of the reinsurer at the end of the contract is the initial reserve plus the reinsurance premium, and the probability of default by the reinsurer is not greater than the risk level of the VaR.

In the first part of the paper, we assume that the insurer wants to determine an optimal reinsurance strategy I^* that maximizes the expected utility of its terminal wealth of $w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I$ under an increasing concave utility function v. That is, we study the following optimization problem:

$$\max_{I \in \mathcal{I}} \mathbb{E}\left[v\left(w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I\right)\right] \quad (1.1)$$

such that $P_I = (1+\theta)\mathbb{E}[I(X)] = p$,

where $0 < p \le (1+\theta)\mathbb{E}(X)$ is a given reinsurance premium budget for the insurer. This optimal reinsurance problem can be viewed as the extension of the classical optimal reinsurance problem without default risk, which was first studied by Arrow (1963) and Borch (1960). As illustrated later in the paper, as $\alpha \to 0$, Problem (1.1) is reduced to the classical optimal reinsurance problem without default risk studied by Arrow (1963) and Borch (1960). We can also recover the solutions of Arrow (1963) and Borch (1960) from our solution to Problem (1.1).

In the second part of the paper, we assume that the insurer wants to use VaR at a risk level $0 < \beta < 1$ to control its total retained risk of $X - I(X) \wedge (\omega_I + P_I) + P_I$ and then seeks an optimal reinsurance strategy I^* that minimizes this VaR. That is, we consider the following optimization problem:

$$\min_{I \in \mathcal{I}} \mathrm{VaR}_\beta\left(X - I(X) \wedge (\omega_I + P_I) + P_I\right). \quad (1.2)$$

This problem is an extension of recent studies on optimal reinsurance under risk measures without default risk such as Balbás et al. (2009), Asimit et al. (2013a,b), Cai et al. (2008), Cheung (2010), Chi (2012), Chi and Tan (2011), and references therein. In particular, and as will be shown later, when $\alpha \le \beta$, Problem (1.2) reduces to the problem without default risk, which was studied by Cheung et al. (2014a,b).

As illustrated in the paper, the solutions to Problems (1.1) and (1.2) are more complicated than those without default risk. Furthermore, the optimal reinsurance strategies in the presence of regulatory initial capital and the counterparty default risk are different both from the optimal reinsurance strategies in the absence of the counterparty default risk and from the optimal reinsurance strategies in the presence of the counterparty default risk but without the regulatory initial capital.

To avoid tedious discussions and arguments, in this paper, we simply assume that the survival function $S_X(x)$ of the underlying loss random variable X is continuous and strictly decreasing on