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A note on multiple life premiums for dependent lifetimes

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ABSTRACT

We study the properties of multiple life annuity and insurance premiums for general symmetric and survival statuses in the case when the joint distribution of future lifetimes has a dependence structure belonging to some nonparametric neighbourhood of independence. The size of the neighbourhood is controlled by a single parameter, which enables us to model really weak as well as stronger dependencies. We provide bounds on the difference of multiple life premiums for vectors of dependent and independent future lifetimes with the same univariate marginal distributions. Each such upper bound can be treated as a premium loading related to the strength of lifetimes' dependence.

1. Introduction

Future lifetimes within a group of people like married couples or family members can exhibit dependencies due to the effect of exposure to some common risk factors, similar lifestyles, genetic diseases or the "broken heart syndrome"; see Parkes et al. (1969), Jagger and Sutton (1991), Martikainen and Valkonen (1996), Hougaard (2000) and Denuit et al. (2001), among others. Recently, a number of works have been devoted to the study of the impact of dependency on actuarial multiple life functions. The classical common shock model of dependent lives, proposed by Marshall and Olkin (1967), as well as Frank's copula model have been discussed in this context by Bowers et al. (1997). Other parametric copula models were analysed by, for example, Carrière and Chan (1986), Frees et al. (1996), Carrière (2000), Denuit et al. (2001) and Spreeuw (2006). The frailty and Markov models, introduced by Norberg (1989) and Oakes (1989), were developed by, e.g., Denuit et al. (2001) and Fulla and Laurent (2008). The influence of positive quadrant dependency, association and right tail monotonicity of lifetimes on actuarial functions and premiums is quantified, among others, by Norberg (1989), Denuit and Cornet (1999), Dhaene et al. (2000) and Denuit et al. (2001). Some relations between premiums of multiple life insurances and annuities for partially ordered vectors of future lifetimes were established by, e.g., Denuit and Lefèvre (1997) and Denuit et al. (1999). Models of stochastic mortality were investigated by, for example, Luciano et al. (2008) and Ma and Yun (2010). A comprehensive survey of the actuarial theory for dependent risks is presented in the monograph by Denuit et al. (2005).

In this paper we study the effect of dependence on multiple life premiums in the case when the dependence structure of future lifetimes is not fully known, but it belongs to some neighbourhood of independence motivated by ψ-mixing (cf. Kaluszka and Okolewski, 2011). In Section 3 we consider general symmetric statuses and give some evaluations on the difference of single annuity premiums as well as on the difference of single insurance premiums calculated for vectors of independent and dependent lifetimes with the same univariate marginal distributions. They are expressed in terms of numerical characteristics of the independent lifetimes. In Section 4 we present similar results for general survival statuses in the case of exchangeable future lifetimes. Each such upper bound can be seen as an additional (in comparison to the case of independent lifetimes) premium loading related to the dependence of lifetimes. It is equal to a coefficient determined by the independent lifetimes multiplied by a coefficient describing the neighbourhood size.

2. Notation

Consider a group of m lives with initial ages $x_1, x_2, \dots, x_m$ and future lifetimes $T_1, T_2, \dots, T_m$. Let $W_j(t) = \mathbf{1}(T_j > t)$ denote the status of the jth life at time t. Here and subsequently, $\mathbf{1}(z) = 1$ if z is true and $\mathbf{1}(z) = 0$ otherwise. By convention, 0 and 1 mean the failure and the intactness, respectively. We will denote by ${}_tp_{x_j}$ the probability that the jth life will survive at least t years. The multiple life status at time t is defined as

$$W_u(t) = u(W_1(t), \dots, W_m(t)),$$

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where the function $u : \{0, 1\}^m \to \{0, 1\}$ is specified in the multiple life contract. We can call u the multiple life status as for given $T_1, \ldots, T_m$ it uniquely determines the process $(W_u(t))_{t\geq 0}$.

We say that a multiple life status u is symmetric if u is a symmetric function, i.e. for any $y_1, \ldots, y_m \in \{0, 1\}$ and any permutation $\pi = (\pi(1), \ldots, \pi(m))$ of the set $\{1, 2, \ldots, m\}$,

$$u(y_1, \ldots, y_m) = u(y_{\pi(1)}, \ldots, y_{\pi(m)}).$$

Examples of symmetric statuses include:

(i) the joint-life status
$$u(y_1, \ldots, y_m) = \min(y_1, \ldots, y_m),$$
denoted in the literature as $x_1 : \ldots : x_m$,

(ii) the last-survivor status
$$u(y_1, \ldots, y_m) = \max(y_1, \ldots, y_m),$$
denoted as $\overline{x_1 : \ldots : x_m}$,

(iii) the exactly k survivors status
$$u(y_1, \ldots, y_m) = \mathbf{1}\left(\sum_{j=1}^{m} y_j = k\right),$$
marked symbolically as $\overline{x_1 : x_2 : \ \ldots \ : x_m}^{[k]}$,

(iv) the kth-survivor status
$$u(y_1, \ldots, y_m) = \mathbf{1}\left(\sum_{j=1}^{m} y_j \geq k\right),$$
marked with the symbol $\overline{x_1 : x_2 : \ \ldots \ : x_m}^{k}$

(cf. Gerber, 1990). The joint-life status and the last-survivor status are special cases of the exactly k survivors status with $k = m$ and $k = 1$, respectively. An example of asymmetric statuses for $m = 2$ is the status $u(y_1, y_2) = \mathbf{1}(y_1 < y_2)$, related to reversionary annuities and denoted by $x_1|x_2$.

3. The general symmetric status

The net single multiple live annuity and insurance premiums for general symmetric statuses can be determined by the use of the Schuette–Nesbitt formulae (cf. Gerber, 1979, 1990), which state that for any reals $c_0, c_1, \ldots, c_m, d_1, \ldots, d_m$ and any events $B_1, \ldots, B_m$,

$$\sum_{k=0}^{m} c_k \cdot \mathsf{P}(N = k) = \sum_{k=0}^{m} \tilde{c}_k \cdot S_k \tag{1}$$

and

$$\sum_{k=1}^{m} d_k \cdot \mathsf{P}(N \geq k) = \sum_{k=1}^{m} \tilde{d}_k \cdot S_k, \tag{2}$$

where $N(\omega) = \sum_{k=1}^{m} \mathbf{1}(\omega \in B_k)$ denotes the number of events that occur, $S_0 = 1$,

$$S_k = \sum_{C(m,k)} \mathsf{P}\left(B_{j_1} \cap \cdots \cap B_{j_k}\right), \quad k = 1, 2, \ldots, m,$$

in which the summation is over the set $C(m, k)$ of all k-element subsets of $\{1, \ldots, m\}$,

$$\tilde{c}_k \equiv \left(\Delta^k c\right)_0 = \sum_{t=0}^{k} c_t (-1)^{k-t} \binom{k}{t} \tag{3}$$

and

$$\tilde{d}_k \equiv \left(\Delta^{k-1} d\right)_1 = \sum_{t=1}^{k} d_t (-1)^{k-t} \binom{k-1}{t-1}. \tag{4}$$

Here, $c = (c_0, c_1, \ldots, c_m, 0, 0, \ldots)$, $d = (0, d_1, \ldots, d_m, 0, 0, \ldots)$ and Δ is the difference operator, i.e. for any sequence $b = (b_0, b_1, b_2, \ldots)$ of real numbers and any $k = 0, 1, 2, \ldots$,

$$(\Delta b)_k = b_{k+1} - b_k, \qquad \Delta^k b = \Delta^{k-1}(\Delta b)$$

and $\left(\Delta^0 b\right)_k = b_k$. An alternative simple proof of the Schuette–Nesbitt formulae is presented in the Appendix.

Denote by ${}_tp_u$ the probability that the multiple life status u is intact at time t. Putting $B_k = \{T_k > t\}$, $S_0(t) = 1$ and

$$S_k(t) = \sum_{C(m,k)} {}_tp_{x_{j_1}:x_{j_2}:\ldots:x_{j_k}}, \quad k = 1, 2, \ldots, m,$$

yields

$$\sum_{k=0}^{m} c_k \cdot {}_tp_{\overline{x_1:x_2:\ldots:x_m}^{[k]}} = \sum_{k=0}^{m} \tilde{c}_k \cdot S_k(t) \tag{5}$$

and

$$\sum_{k=1}^{m} d_k \cdot {}_tp_{\overline{x_1:x_2:\ldots:x_m}^{k}} = \sum_{k=1}^{m} \tilde{d}_k \cdot S_k(t). \tag{6}$$

From (5) we obtain, e.g., the following identity for the net single continuous whole life annuity premium for the general symmetric status

$$\sum_{k=0}^{m} c_k \cdot \bar{a}_{\overline{x_1:x_2:\ldots:x_m}^{[k]}} = \sum_{k=0}^{m} \tilde{c}_k \cdot S_k^{\bar{a}}, \tag{7}$$

where

$$S_k^{\bar{a}} = \sum_{C(m,k)} \bar{a}_{x_{j_1}:x_{j_2}:\ldots:x_{j_k}}$$

and

$$\bar{a}_u = \int_0^{\infty} v^t \, {}_tp_u \, dt, \tag{8}$$

in which v denotes the discount factor.

The above formulae show that the general symmetric status premium validation problem resolves itself into determining the probabilities ${}_tp_{x_{j_1}:x_{j_2}:\ldots:x_{j_k}}$. If $T_1, \ldots, T_m$ are independent random variables, then

$${}_tp_{x_{j_1}:x_{j_2}:\ldots:x_{j_k}} = \prod_{i=1}^{k} {}_tp_{x_{j_i}}.$$

One can proceed analogously if the dependence structure (the copula) of $(T_1, \ldots, T_m)$ is completely known (see, e.g. Bowers et al., 1997, for $m = 2$). For example,

$${}_tp_{x_{j_1}:x_{j_2}:\ldots:x_{j_k}} = \left(\prod_{i=1}^{k} {}_tp_{x_{j_i}}\right)\left[1 + \alpha \sum_{1\leq i<l\leq k} {}_tq_{x_{j_i}} \, {}_tq_{x_{j_l}}\right], \tag{9}$$

where ${}_tq_x = 1 - {}_tp_x$, provided that $(T_1, \ldots, T_m)$ has the following Farlie–Gumbel–Morgenstern m-copula

$$C(u_1, \ldots, u_m) = \left(\prod_{i=1}^{m} u_i\right)\left[1 + \alpha \sum_{1\leq i<j\leq m} (1 - u_i)(1 - u_j)\right], \tag{10}$$

in which α is a bounded parameter describing the strength of dependence (see Kotz et al., 2000).

If $(T_1, \ldots, T_m)$ are dependent but their copula is not fully known, then we propose to assume that this copula belongs to some neighbourhood of independence, e.g. that for some $\varepsilon > 0$ and all $1 \leq k_1 < \cdots < k_i \leq m$,

$$\sup_{x>0} \frac{\left|\mathsf{P}(T_{k_1} > x, \ldots, T_{k_i} > x) - \mathsf{P}(T_{k_1} > x) \ldots \mathsf{P}(T_{k_i} > x)\right|}{\mathsf{P}(T_{k_1} > x) \ldots \mathsf{P}(T_{k_i} > x)} \leq \varepsilon. \tag{11}$$