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Reserve-dependent benefits and costs in life and health insurance contracts

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ABSTRACT

Premiums and benefits associated with traditional life insurance contracts are usually specified as fixed amounts in policy conditions. However, reserve-dependent surrender values and reserve-dependent expenses are common in insurance practice. The famous Cantelli theorem in life insurance ensures that under appropriate assumptions surrendering can be ignored in reserve calculations provided that the surrender payment equals the accumulated reserve. In this paper, more complex reserve-dependent payment patterns are considered, in line with insurance practice. Explicit formulas are derived for the corresponding reserve.

1. Introduction

Multistate models provide a convenient representation for generalized life insurance contracts, including life insurance policies, disability insurance policies and permanent health insurance policies, for instance. Each state represents a particular status for the policyholder. The benefits comprised in the contract are associated to sojourns in, or transitions between states. See, e.g., Chapter 8 in Dickson et al. (2009) for an introduction.

Under the Markovian assumption, Thiele's differential equation describes the dynamics of the accumulated reserve. As it can easily be solved numerically, using Euler's method for instance, it provides an efficient tool to perform actuarial calculations. The situation becomes nevertheless more difficult when benefits are expressed in terms of the reserves, as in the case of surrendering for instance. The famous Cantelli theorem ensures that under appropriate conditions surrendering can be ignored in the reserve calculations provided that the surrender payment equals the reserve. This is true from a prospective perspective as well as from a retrospective perspective. However, the insurer generally applies a penalty when the policyholder cancels the contract so that this result is of little practical use.

In this paper, we consider reserve-dependent payment patterns and we derive explicit expressions for the reserve. Typical examples of reserve-dependent insurance benefits include:

- Surrender payments, with the surrender value equal to the accumulated reserve minus a cancellation fee.
- Capital management fees proportional to the reserves.
- Profit participation, with surplus dividends depending on the accumulated reserve.

We show that, under fairly general conditions, one can still apply Cantelli's theorem to derive an explicit expression for the reserves provided that the structure of the benefits and premiums is appropriately modified. Several examples are discussed to illustrate the applicability of the approach proposed in the present paper.

The topic investigated here has already been examined in the literature. For instance, Norberg (1991) studies general multistate life insurance products and points out to the fact that Thiele's differential equation can also cope with payments depending on the reserves in a linear way. This author derives explicit expressions for the accumulated reserves in two particular cases:

(a) a widow's pension where the retrospective reserve is paid back to the husband in case the wife dies first, and
(b) a widow's pension with administration expenses expressed as a linear function of the reserve.

The present paper expands on the ideas of Norberg (1991) and presents explicit expressions for more general contracts. Milbrodt and Helbig (1999) also mention the key role played by Thiele's

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equations if benefits are reserve dependent. They discuss an annuity insurance with flexible time of retirement and death benefits and calculate accumulated reserves when surrender payments equal a fixed proportion of the accumulated reserve.

Notice that the problem considered here has also been discussed in several textbooks. For instance, in the multiple decrement model, Bowers et al. (1997, Section 11.4) show that as long as the withdrawal benefit in a double decrement model whole life insurance is equal to the reserve under the associated single decrement model, premiums and reserves coincide under the single and double decrement models. The present paper revisits this problem in a more general framework. It is shown that the conclusion drawn e.g. by Bowers et al. (1997) can be generalized, provided that mild conditions are fulfilled.

The remainder of this paper is organized as follows. Section 2 briefly recalls the multistate Markovian setting for describing generalized life insurance contracts. In particular, definitions for the prospective and retrospective reserves are provided. Section 3 is devoted to Cantelli's fundamental theorem, which provides the technical argument used in Section 4 to derive the results in the case of reserve-dependent insurance payments. Section 5 discusses several examples of practical relevance before the final Section 6 concludes the paper.

2. Multistate life insurance

Consider a k-state Markov transition model describing some insurance contract. The initial state is numbered 1. The policyholder is aged x at policy issue and time t measures the seniority of the contract, $t = 0$ corresponding to policy issue. The transition intensity functions are indexed by attained age and denoted as $\mu_{x+t}^{(ij)}$ for different states $i \neq j \in \{1, 2, \ldots, k\}$. All transition intensities have to be integrable functions. We define

$$\mu_{x+t}^{(ii)} = -\sum_{j:j\neq i} \mu_{x+t}^{(ij)}.$$

The corresponding transition probabilities between states i and j over the time interval (s, t) are denoted as ${}_{t-s}p_{x+s}^{(ij)}$. Clearly,

$${}_{t-s}p_{x+s}^{(ii)} = 1 - \sum_{j:j\neq i} {}_{t-s}p_{x+s}^{(ij)}.$$

These probabilities can be obtained as the unique solution of Kolmogorov's backward equations

$$\frac{\mathrm{d}}{\mathrm{d}s} {}_{t-s}p_{x+s}^{(ij)} = -\sum_{l\in\{1,2,\ldots,k\}} {}_{t-s}p_{x+s}^{(il)}\, \mu_{x+s}^{(lj)}$$

with initial condition ${}_0p_{x+s}^{(ij)} = 0$ for $i \neq j$ and ${}_0p_{x+s}^{(ii)} = 1$. Provided that the transition intensity functions are piecewise constant, the Cox–Miller formula gives the explicit solution to this system. Likewise the transition probabilities also uniquely solve Kolmogorov's forward equations.

The interest earned on the savings account is modeled by the cumulative interest intensity function Δ_t. We assume that Δ_t has finite variation on compacts. Usually we have $\mathrm{d}\Delta_t = \delta_t \mathrm{d}t$ for some interest intensity function δ_t. The corresponding discount factor ${}_{t-s}v_s$ is the unique solution of

$${}_{t-s}v_s = 1 - \int_{(s,t]} {}_{t-u}v_u\, \mathrm{d}\Delta_u.$$

Let $c^{ij}(t)$ be the benefit paid by the insurer upon a transition from state i to state $j \neq i$ occurring at time $t \in (0, n]$. We assume that the functions $t \mapsto c^{ij}(t)$ are Borel-measurable and bounded. Let $\mathrm{d}B^i(t)$ be the sojourn benefit (net of premiums) in state i at time $t \in [0, n]$. Then $B_i(t)$ describes the accumulated sojourn payments (minus premiums paid) in state i in the time interval $[0, t]$. We assume that the functions $t \mapsto B^i(t)$ have finite variation.

The appropriate definition of the reserves has been investigated by Wolthuis and Hoem (1990). The prospective reserve at time t in state i is clearly given by

$$V_+^i(t) = \sum_{j=1}^k \int_{(t,n]} {}_{s-t}v_t\; {}_{s-t}p_{x+t}^{(ij)} \mathrm{d}B^j(s) + \sum_{\substack{j,l=1\\ j\neq l}}^k \int_t^n {}_{s-t}v_t\; {}_{s-t}p_{x+t}^{(ij)} \mu_{x+s}^{(jl)} c^{jl}(s)\mathrm{d}s. \quad (1)$$

The reserve (1) can also be obtained as the solution of Thiele's differential equation, which is given by

$$\mathrm{d}V_+^i(t) = V_+^i(t-)\mathrm{d}\Delta_t - \mathrm{d}B^i(t) - \sum_{\substack{j=1\\ j\neq i}}^k \mu_{x+t}^{(ij)} \big(c^{ij}(t) + V_+^j(t) - V_+^i(t)\big)\mathrm{d}t \quad (2)$$

with terminal condition $V_+^i(n) = 0$, $i = 1, \ldots, k$. Milbrodt and Helbig (1999) show uniqueness of the solution of (2), but only for payment functions that do not depend on the reserve. In Section 4, we provide a uniqueness result also for reserve-dependent payments, given that the dependence is linear.

Provided that the transition matrix $\boldsymbol{P}(s, t) = ({}_{t-s}p_{x+s}^{(ij)})_{i,j=1,\ldots,k}$ is regular, the retrospective reserve is defined according to Wolthuis and Hoem (1990) as

$$V_-^i(t) = -\sum_{j=1}^k \int_{[0,t]} ({}_{t-s}v_s)^{-1} (\boldsymbol{P}(s, t))_{ij}^{-1} \mathrm{d}B^j(s) - \sum_{\substack{j,l=1\\ j\neq l}}^k \int_0^t ({}_{t-s}v_s)^{-1} (\boldsymbol{P}(s, t))_{ij}^{-1} \mu_{x+s}^{(jl)} c^{jl}(s)\mathrm{d}s \quad (3)$$

where $(\boldsymbol{P}(s, t))_{ij}^{-1}$ is element (i, j) of the inverse of the transition matrix $\boldsymbol{P}(s, t)$. It can also be obtained as the solution of Thiele's equation (2) recalled above but with initial condition $V_-^i(0-) = 0$ and V_+^i replaced by V_-^i.

Formulas (1) and (3) are true regardless of whether the payment functions are reserve dependent or not. However, in the case of reserve-dependent payments, (1) and (3) only implicitly describe the prospective and the retrospective reserves, respectively, and it is more convenient to work with Thiele's equation (2).

In order to calculate prospective and retrospective reserves, interest and transition intensity functions have to be chosen. Note that in the retrospective view the interest and transition intensity functions relate to the past and, thus, their realized values can be observed. This implies that the observed basis can be used as basis for the retrospective calculations. In the prospective view, however, interest and transition intensities relate to the future and therefore the prospective calculations are always performed with an assumed basis. Under this assumed basis, the values of retrospective and prospective reserves are equal provided that premiums are determined by the equivalence principle (using the same basis). Let us mention that the International Association of Insurance Supervisors (IAIS) recommends that the liabilities of insurance companies should be evaluated on consistent bases, i.e. by means of an economic valuation that reflects the prospective future cash flows. In Europe, Solvency II determines that the best estimate of the provision for future commitments must be measured based on current information and realistic predictions.

3. Cantelli's theorem

The results in this section are true for both retrospective and prospective reserves. In the remainder of this paper, we write $V^i(t)$ if a statement is true for both $V_-^i(t)$ and $V_+^i(t)$.