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Analytic solution for ratchet guaranteed minimum death benefit options under a variety of mortality laws

Eric R. Ulm *

Department of Risk Management and Insurance, J. Mack Robinson College of Business, Georgia State University, Atlanta, GA, United States

HIGHLIGHTS

- We derive a number of analytic results for GMDB ratchet options.
- Closed form solutions are found for simple mortality laws.
- We find an infinite series solution for a general mortality laws.
- We derive the conditions under which this series terminates.
- We sum this series for at-the-money options under the realistic Makeham's Law.

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ABSTRACT

We derive a number of analytic results for GMDB ratchet options. Closed form solutions are found for De Moivre's Law, Constant Force of Mortality, Constant Force of Mortality with an endowment age and constant force of mortality with a cutoff age. We find an infinite series solution for a general mortality laws and we derive the conditions under which this series terminates. We sum this series for at-the-money options under the realistic Makeham's Law of Mortality.

1. Introduction

Much work has been done on the valuation of Guaranteed Minimum Death Benefits (GMDB) embedded in variable annuities contracts since their introduction over 20 years ago. In a variable annuity contract, the policyholder invests in an equity-based account and receives the account value at maturity of the contract which he can then chose to annuitize if he desires. GMDB riders guarantee some minimum benefit at death, which is usually at least the return of premiums paid. It is sometimes the case that the contract guarantees that the beneficiary receives the maximum value the contract has attained, a situation commonly called a "ratchet" GMDB although it is analogous to lookback options in the financial literature. Because the annuities are invested in stock funds, GMDB riders resemble a sequence of lookback put options, with the put value on a given date being multiplied by the probability the annuitant dies that date and has not yet lapsed his policy. The formula for a floating strike lookback put was originally derived by Gatto et al. (1979) and can be found in Haug (2007) pg. 142. One can integrate this function with respect to a probability distribution of maturity times in order to numerically evaluate the GMDB, see Hardy (2003). Brennan and Schwartz (1976) was the first paper to price contracts involving a combination of equity guarantees and mortality rates.

A number of papers address issues specific to GMDB pricing, including Milevsky and Posner (2001) and Milevsky and Salisbury (2001), which derives a differential equation that must be satisfied by the GMDB value. A number of solutions have been found assuming constant mortality and lapse for return of premium GMDBs, both in that paper and by Ulm (2006). Ulm (2008) extends these results to roll-up GMDBs as well as finding analytic solutions for De Moivre's Law of mortality. This paper also derives results for

* Tel.: +1 404 413 7485.
E-mail address: eulm@gsu.edu.

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policies that endow at a certain age as well as those whose guarantees are eliminated after a certain age. Gerber et al. (2012) derive results for a number of basic options in addition to simple puts. These studies all have the drawback that the mortality laws used are not very representative of real human mortality. In this paper, we derive some analytic results on ratchet options, including results on the more realistic Makeham Law of Mortality. This is a valuable contribution as there is a significant need in practice for fast and accurate methods of calculation. Milevsky (2006) pg. 259 mentions that "it is very difficult to obtain a closed-form solution" for GMDB options and to date not many exist.

We determine these values by solving the P.D.E. derived by Milevsky and Salisbury (2001) using Laplace Transform techniques. Transform techniques have been used to solve similar PDEs. Davydov and Lintesky (2001) price continuous barrier and lookback options, Petrella and Kou (2006) price discrete barrier and lookback options and Zhu and Lian (2011) price variance swaps using transforms.

2. The differential equation to be solved

Assume the existence of a deferred annuity with a variable account. The annuity has a GMDB rider that guarantees a return of premium upon the death of the annuitant. This could be modeled as the sum of a continuous sequence of European put options (see, for example, Hardy (2003)). The weight at a given option duration would be equal to the instantaneous probability of death at that moment. The value of the strike at that moment would be X, the value of the initial premium paid. The equation that must be satisfied by the value of the GMDB, if $S \le Xf_a(S,t)$, is:

$$\frac{\partial f_a}{\partial t} + [r(t) - q]S\frac{\partial f_a}{\partial S} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 f_a}{\partial S^2} = [r(t) + \mu_x(t) + \lambda(S,t)]f_a - [\mu_x(t)](X - S) \quad (1)$$

which is derived in Milevsky and Salisbury (2001), or Ulm (2006).

The boundary condition for a ratchet option becomes:

$$\left.\frac{\partial f}{\partial S}\right|_X = \frac{f}{X}. \quad (2)$$

That is, the derivative is continuous across the boundary and the value is equal to $f(X)\frac{S}{X}$ when $S > X$. In the remainder of Section 2, we will avoid making any assumptions on the form of $\mu_x(t)$. We will, however, assume a constant lapse rate of $\lambda(S,t) = \lambda$ as well as a constant risk-free rate and volatility.

2.1. General solution to Eq. (1)

Following Ulm (2008), I will assume $f_a(S,t)$ is of the form:

$$f_a(S,t) = [XA(t) - SB(t)] + C(S,t) \quad (3)$$

which leads to:

$$A'(t) - (r + \mu(t) + \lambda)A(t) = -\mu(t) \quad (4)$$

and

$$B'(t) - (q + \mu(t) + \lambda)B(t) = -\mu(t) \quad (5)$$

as in Ulm. Bowers et al. (1997, page 125) state that

$$\frac{d}{dx}\bar{A}_x - (\delta + \mu(x))\bar{A}_x = -\mu(x) \quad (6)$$

where δ is the force of interest. This implies that $A(t)$ and $B(t)$ are $\bar{A}_t$ at force of interest $r+\lambda$ and $q+\lambda$ respectively.

C must now obey the following equation:

$$\frac{\partial C}{\partial t} - [r + \mu_x(t) + \lambda]C + (r-q)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2} = 0 \quad (7)$$

subject to the boundary condition:

$$\left.\frac{\partial C}{\partial S}\right|_X = \frac{C}{X} + A(t). \quad (8)$$

We now make Eqs. (7) and (8) dimensionless. The derivation follows similar lines to that found in Wilmott et al. (1995) and Ulm (2008). We pick dimensionless variables $y = \ln\left(\frac{S}{X}\right)$ and $\tau = \frac{\sigma^2(T-t)}{2}$. T is currently an arbitrary parameter as, unlike the case for the vanilla European options, there is no expiration date to the GMDB option. Later, we will see that T represents the age at which the GMDB must be exercised. For a mortality function such as De Moivre's law with a built in maximum age, T represents the time remaining until that age is reached. For a mortality function without a maximum age, T can be viewed as the time until the GMDB is no longer a death benefit, but an endowment benefit. If there is no endowment age, we will let $T \to \infty$ as the final step. We will assume $C(S,t) = Xe^{\alpha y}f(\tau)w(y,\tau)$:

We chose α to be $\frac{1}{2} - \frac{(r-q)}{\sigma^2}$ and assume that $f(\tau)$ satisfies:

$$f'(\tau) + \left[\frac{2(r + \mu_x(\tau) + \lambda)}{\sigma^2} + \alpha^2\right]f(\tau) = 0 \quad (9)$$

this leads to the following equation for $w(y,\tau)$:

$$\frac{\partial w}{\partial \tau} - \frac{\partial^2 w}{\partial y^2} = 0. \quad (10)$$

Subject to the boundary condition:

$$\left.\frac{\partial w}{\partial y}\right|_0 = [1-\alpha]w + \frac{A(\tau)}{f(\tau)} \quad (11)$$

and initial condition $w(y,0) = 0$.

The ordinary differential equation for $f(\tau)$ is solved by:

$$f(\tau) = e^{-\kappa(\tau-\tau_0)}e^{-\frac{2}{\sigma^2}\int_{\tau_0}^{\tau}\mu(s)ds} \quad (12)$$

where $\kappa = \frac{2(r+\lambda)}{\sigma^2} + \alpha^2$, τ_0 is an arbitrary constant, and $\mu(s)$ is the functional form of $\mu(\tau)$ not $\mu(t)$.

Eq. (10) and associated boundary conditions can be solved by finding the Laplace transform relative to τ:

$$pg(p) - \frac{\partial^2 g}{\partial y^2} = w(y,0) = 0. \quad (13)$$

With transformed boundary condition:

$$\left.\frac{\partial g}{\partial y}\right|_0 = [1-\alpha]g + A^*(p) \quad (14)$$

$$A^*(p) = L\left[\frac{A(\tau)}{f(\tau)}\right]. \quad (15)$$

Satisfying the boundary conditions and the requirement that the function stay finite as $y \to \infty$ gives:

$$g(y,p) = \frac{A^*(p)}{\sqrt{p} + \alpha - 1}e^{y\sqrt{p}}. \quad (16)$$

We now turn our attention to finding $A^*(p)$ for specific mortality functions and inverting the Laplace transform.

3. Solution to Eq. (1) under specific mortality laws

3.1. The value of a Ratchet GMDB under De Moivre's law of mortality

The general procedure for determining the value of the option is to find $A(t)$, $B(t)$ and, find the function in Eq. (16) and invert the