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Pricing and hedging of variable annuities with state-dependent fees

Łukasz Delong *

Institute of Econometrics, Division of Probabilistic Methods, Warsaw School of Economics, Niepodległości 162, 02-554 Warsaw, Poland

HIGHLIGHTS

- We investigate variable annuity contracts for which the fee deducted from the policyholder's account depends on the account value.
- We consider an incomplete financial market modelled with a two-dimensional Lévy process.
- We apply a quadratic pricing and hedging objective.
- We derive an equation from which the fee for the maturity guaranteed benefit can be calculated.
- We characterize a strategy which allows the insurer to hedge the maturity guaranteed benefit.

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ABSTRACT

We investigate the problem of pricing and hedging variable annuity contracts for which the fee deducted from the policyholder's account depends on the account value. It is believed that state-dependent fees are beneficial to policyholders and insurers since they reduce policyholders' incentives to lapse the policies and match the costs incurred by policyholders with the pay-offs received from embedded guarantees. We consider an incomplete financial market which consists of two risky assets modelled with a two-dimensional Lévy process. One of the assets is a security which can be traded by the insurer, and the second asset is a security which is the underlying fund for the variable annuity contract. In our model we derive an equation from which the fee for the guaranteed benefit can be calculated and we characterize a strategy which allows the insurer to hedge the benefit. To solve the pricing and hedging problem in an incomplete financial market we apply a quadratic objective.

1. Introduction

Variable annuities are among the most popular insurance contracts sold worldwide. Their popularity is due to the fact that variable annuities combine insurance with investment by providing a protection against life contingencies and a participation in the growth of the financial market. Variable annuities provide benefits which are contingent on the performance of investment funds together with capital protections which guarantee a minimum rate of return from the investment. Nowadays, we find a range of variable annuity contracts which guarantee a minimum death benefit, minimum maturity benefit, minimum income benefit and minimum accumulation benefit.

The problem of pricing and hedging variable annuities has been thoroughly studied in the actuarial literature, see among others Bacinello et al. (2011), Bauer et al. (2008), Bernard et al. (2014), Coleman et al. (2007), Deelstra and Rayée (2013), Hardy (2003), Quittard-Pinon and Kelani (2013). From the financial point of view the capital protection embedded in a variable annuity is a financial option on an investment fund. Consequently, techniques from financial mathematics should be applied in order to price and hedge variable annuity benefits. However, there is a significant difference between pricing and hedging financial options and guarantees embedded in variable annuities. A financial option is financed with a premium which is paid by the buyer of the option at the inception of the contract, whereas a guarantee embedded in a variable annuity is financed with fees which are paid by the policyholder during the lifetime of the contract. Moreover, the fees are deducted from the policyholder's account and those fees should finance the guarantee which is contingent on the policyholder's account value. Those subtle issues, typical for variable annuities, should be reflected in a model which is used for pricing and hedging of variable annuities.

In most variable annuity contracts insurers deduct fees which are proportional to the policyholder's account value. Consequently,

* Tel.: +48 22 5649256.
E-mail address: lukasz.delong@sgh.waw.pl.

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if the account value is low the fee is low, and if the account value is high the fee is high. It has been noticed that such a fee payment scheme increases incentives among insured persons to lapse their policies. Guarantees which are embedded in variable annuities are similar to put options, which means that the guarantee is in-the-money if the account value is low and is out-of-the money if the account value is high. If a proportional fee is deducted from the account, then the policyholder pays a high fee for the guarantee in times when the guarantee is not valuable to him. Clearly, the policyholder is not satisfied if he has to pay a lot of money for the embedded guarantee which he does not need in times of growing economy and, consequently, he is very likely to lapse the policy. In order to reduce policyholders' incentives for lapsing variable annuities it has been suggested that state-dependent fees should be introduced by insurers. In recent years Prudential UK introduced a variable annuity with a guaranteed minimum return under which the fee is deducted from the account at a fixed rate only if the account value is below a guaranteed level. Under such an account-dependent payment scheme the fee for the guarantee is paid only if the guarantee is valuable to the policyholder. The advantage of such a fee payment scheme is that it reduces policyholders' incentives to lapse the policies and matches the costs incurred by policyholders with the pay-offs received from embedded guarantees, but the disadvantage is that the insurer who collects the fee only in times when the guarantee is in-the-money must set the fee rate at a level which is higher than the constant fee rate.

A variable annuity contract with a fee which is deducted at a fixed rate only if the account value is below a pre-specified level has been recently studied in Bernard et al. (in press). The authors consider a complete Black–Scholes financial model with one risky asset and derive an equation from which the fee for the embedded guarantee can be calculated. The problem of hedging the guaranteed benefit is not considered in Bernard et al. (in press). In fact, the hedging strategy in the model from Bernard et al. (in press) is trivial since the authors consider a complete financial market and, consequently, the delta-hedging strategy (the replicating strategy) is the only hedging strategy which can be used. To the best of our knowledge the paper by Bernard et al. (in press) is the only paper in the literature which studies variable annuities with state-dependent fees. Our paper is the second one in this field. We would like to point out that our financial model and our pricing and hedging problem are more general than the model and the problem from Bernard et al. (in press).

In this paper we consider an incomplete financial market which consists of two risky assets modelled with a two-dimensional Lévy process. One of the assets is a security which can be traded by the insurer, and the second asset is a security which is the underlying fund for the variable annuity contract. Hence, in this paper we take into account two important sources of market incompleteness which the insurer must face in reality. The first source of market incompleteness comes from unpredictable jumps (crashes) in the asset price which are modelled with a discontinuous Lévy process, and the second source of market incompleteness comes from the impossibility to trade the fund on which the variable annuity is contingent. We would like to point out that in reality the insurer can never trade the underlying fund (an exotic external fund) for a variable annuity and asymmetric heavy tails of asset returns and crashes in the market are the main financial risks for the insurer selling a variable annuity. As far as the fee payment scheme is concerned, which is the crucial point in our paper, we consider a general state-dependent fee which is modelled as a function of the account value. Our fee process includes the fee process considered in Bernard et al. (in press). To solve the pricing and hedging problem in our incomplete financial model we apply a quadratic objective and we require that the mismatch between the hedging portfolio and the liability at the terminal time is minimal in a mean-square sense. We derive an equation from which the fee for the guaranteed benefit can be calculated and we find the hedging strategy which allows the insurer to hedge optimally the benefit. We use a backward stochastic differential equation to characterize the fee and the hedging strategy. We point out that quadratic pricing and hedging is very popular in financial mathematics and we would like to mention recent papers by Ankirchner and Heine (2012), Fujii and Takahashi (in press), Jeanblanc et al. (2012), Kharroubi et al. (2013), and Kohlmann et al. (2010) where backward stochastic differential equations are used.

This paper is structured as follows. In Section 2 we describe the model. In Section 3 we solve a quadratic optimization problem and in Section 4 the solution of the quadratic optimization problem is used to solve the pricing and hedging problem for variable annuities with state-dependent fees. In Section 5 we present a numerical example which illustrates how our solution can be applied in practice. In the numerical example the dependence between Lévy processes is modelled with a Lévy Clayton copula.

2. The model

We deal with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_t)_{0\le t\le T}$ and a finite time horizon $T < \infty$. We assume that $\mathcal{F}$ satisfies the usual hypotheses of completeness ($\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we define an $\mathcal{F}$-adapted, two-dimensional Lévy process $L = (L_F, L_S) = (L_F(t), L_S(t), 0 \le t \le T)$. Its discontinuous part is denoted by $L^d = (L_F^d, L_S^d)$.

The financial market consists of a risk-free bank account $R = (R(t),\ 0 \le t \le T)$ and two risky assets $F = (F(t),\ 0 \le t \le T)$ and $S = (S(t),\ 0 \le t \le T)$. The value of the risk-free bank account satisfies the dynamics

$$R(t) = R(0)e^{rt}, \quad 0 \le t \le T, \tag{2.1}$$

and the prices of the risky assets are modelled with dependent exponential Lévy processes, i.e. they satisfy the dynamics

$$F(t) = F(0)e^{L_F(t)}, \qquad S(t) = S(0)e^{L_S(t)}, \quad 0 \le t \le T.$$

By the Lévy–Itô decomposition, see Theorem 2.4.1 in Applebaum (2004), we can consider the representations

$$\begin{aligned} F(t) &= F(0)e^{\mu_F^* t+\sigma_{F,1}W(t)+\sigma_{F,2}B(t)+\int_0^t\int_{\mathbb{R}^2} z_F\tilde{N}(ds,dz_F,dz_S)}, \\ &\quad 0 \le t \le T, \\ S(t) &= S(0)e^{\mu_S^* t+\sigma_{S,1}W(t)+\sigma_{S,2}B(t)+\int_0^t\int_{\mathbb{R}^2} z_S\tilde{N}(ds,dz_F,dz_S)}, \\ &\quad 0 \le t \le T, \end{aligned} \tag{2.2}$$

where $W = (W(t),\ 0 \le t \le T)$ and $B = (B(t),\ 0 \le t \le T)$ are independent Brownian motions, and N is a random measure on $\Omega \times \mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{R}^2)$ which is independent of (W, B). The compensated random measure $\tilde{N}$ is defined by

$$\tilde{N}(dt, dz_F, dz_S) = N(dt, dz_F, dz_S) - \nu(dz_F, dz_S)dt,$$

where ν is a σ-finite measure on $\mathcal{B}(\mathbb{R}^2)$ called a Lévy measure. We set $N([0, T], \{(0, 0)\}) = \nu(\{(0, 0)\}) = 0$ and we assume that

(A) $\int_{\mathbb{R}^2}(e^{2z_F} + e^{2z_S})\nu(dz_F, dz_S) < \infty$.

The random measure N counts the number of jumps of a given size of the Lévy process $L = (L_F, L_S)$, see Chapter 2.3 in Applebaum (2004). We point out that we use dependent Lévy process (L_F, L_S) to model the asset prices (F, S). The continuous parts of the Lévy processes are correlated with coefficient ρ. Hence, by the Cholesky decomposition we can choose

$$\sigma_{S,1} = \sigma_S, \qquad \sigma_{S,2} = 0,$$

$$\sigma_{F,1} = \sigma_F\rho, \qquad \sigma_{F,2} = \sigma_F\sqrt{1-\rho^2}. \tag{2.3}$$