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Factor risk quantification in annuity models

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HIGHLIGHTS

- We examine risk capital allocation methods on annuity prices.
- We apply approximations to annuity model to transform it to a linear loss model.
- We determine interest rate risk and mortality risk contributions to the total loss.
- Factor risk contributions under both approximations are similar.
- Suggested approximations are useful and work efficiently.

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ABSTRACT

Calculation of risk contributions of sub-portfolios to total portfolio risk is essential for risk management in insurance companies. Thanks to risk capital allocation methods and linearity of the loss model, sub-portfolio (or position) contributions can be calculated efficiently. However, factor risk contribution theory in non-linear loss models has received little interest. Our concern is the determination of factor risk contributions to total portfolio risk where portfolio risk is a non-linear function of factor risks. We employ different approximations in order to convert the non-linear loss model into a linear one. We illustrate the theory on an annuity portfolio where the main factor risks are interest-rate risk and mortality risk.

1. Introduction

Financial risk management is mainly concerned with identification, quantification (or allocation) and management of risk drivers of portfolios (or companies). Determination of the main risk drivers and calculation of the risk driver contributions to overall portfolio risk is essential for accurate risk management. Together with right risk allocation methods and appropriate risk measures, risk managers can create value for the companies. Once the risk capital of the portfolio is calculated (based on a chosen risk measure), the risk manager can determine sub-portfolios contributions or factor risk contributions. The former one stands for the sub-portfolios, instruments or constituents, whereas the latter one stands for the factors affecting portfolio losses such as interest rates and mortality rates.

Sub-portfolio contributions are of great importance for: management decision support and business planning, performance measurement and risk-based compensation, pricing, profitability assessment and limits, building optimal risk-return portfolios and strategies, see Alexander (2004), Dhaene et al. (2012). In this case, the overall portfolio loss is a linear combination of sub-portfolio losses, and allocation methods for this type of problems have been well studied in the literature; see for example, Merton and Perold (1993), Saita (1999), Froot and Stein (1998), Tasche (1999), Cummins (2000), Myers and Read (2001), Denault (2001), Urban et al. (2004), Kim and Hardy (2008), Buch et al. (2009), Karabey (2012b). These authors mainly

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study different allocation methods that can be applied to linear loss models. For determination of sub-portfolio contributions under linear loss models, the marginal allocation (or Euler's allocation) method has been suggested by several authors for different reasons:

- Tasche (1999) shows that Euler's principle is compatible with portfolio optimization/performance measurement for a positive homogeneous and differentiable risk measure.
- Denault (2001) derived Euler's allocation principle by game theory which was regarded as the Aumann–Shapley allocation principle. He argues that allocation based on that method is the unique fair allocation principle for a coherent risk measure.
- Myers and Read (2001) argue that in order to determine line by line surplus requirements effectively in an insurance company, the most appropriate way is to apply Euler's principle.
- Kalkbrener (2005) argues that Euler's principle is the only allocation principle that is compatible with the diversification effects. Recall that diversification plays an important role for the portfolio management and it is provided by the subadditivity of the risk measures.

Another important problem in risk management is the quantification of factor risk contributions. These contributions provide an understanding of risk sources in the portfolios. However, factor risk contributions theory has received comparatively little interest. There is a challenge around the calculation of factor risk contributions to the total portfolio risk, as the total portfolio risk cannot generally be written as a linear function of separate factor risks. Therefore, the allocation methods for linear loss models cannot be directly applied to non-linear loss models. Recently some authors consider the problem of factor risk contributions. Cherny and Madan (2006) described position contributions of conditional losses given the factor risks. Tasche (2009) investigated the application of Euler's theorem for the identification of the contributions of underlying names to expected losses of collateralized debt obligation (CDO) tranches. He also studied measurement of the impact of systematic factors on portfolio risk. Most recently, Rosen and Saunders (2010) employed the Hoeffding decomposition for the determination of the factor risk contributions to credit risk of a portfolio. As it can be seen above, factor risk contributions theory has been mainly studied on credit risk.

The purpose of the present paper is to take a more detailed look at the factor risk contribution theory in annuity portfolios. In recent decades, life expectancy has improved throughout the world and it has been observed that mortality is a stochastic process in which longevity improvements are unpredictable, see Cairns et al. (2006). It is known that these improvements have greater effects on higher ages which directly cause annuity providers to incur losses on their annuity business. The main problem is that pensioners are living longer than was anticipated. Thus, annuity payments last longer than was anticipated. As a result annuity providers have to bear these costs. Moreover, there is a considerable uncertainty regarding the future development of life expectancy. Thus, the insurers need for risk management of annuity business increases. On the other hand, the regulators have long been focused on the risk in financial investments, however recently, quantification and management of the risk in pension liabilities has become more and more important. The financial regulations of insurance companies in the EU have been redesigned by the Solvency II project, increasing the importance of valuation and management of pension liabilities. By considering the increasing need of risk management/risk quantification in pension funds, we examine factor risk contributions theory on life annuities.

In this paper, we investigate possible approximations for calculation of marginal factor risk contributions under a non-linear annuity model. We apply two approximations that can be used in linearization of the non-linear annuity model: the Hoeffding decomposition and a first order Taylor expansion. We then define the contributions of factor risks under this linear approximation. Furthermore, we study the variance decomposition which is the most commonly used approach for risk decomposition in life insurance modelling, and we compare results.

The paper is organized as follows. Section 2 briefly introduces possible approximations for linearization of a non-linear loss variable. Section 3 describes the future annuity distributions including interest rate and mortality models that we use. We also briefly define bond pricing methodology, loss variables and financial risk measures for the annuity model. Furthermore, we introduce factor risk contributions under mentioned approximations in this section. Sections 4 and 5 present a case study applying the presented theory. In Section 6 we conclude.

2. Risk allocation approximations

We now introduce different methods that enable us to allocate the total risk into factor risks. In this section we briefly describe these approaches for a general loss variable whereas in Sections 3.5 and 3.6 we study these methods for an annuity loss variable in detail.

2.1. The Hoeffding decomposition

The Hoeffding decomposition is an orthogonal decomposition that is used in mathematical statistics to study the distribution of statistics $g(Z_1, \ldots, Z_k)$ for a random sample, that is, of a set of independent random variables $Z_1, \ldots, Z_k$, see Hoeffding (1948). In a mathematical sense a general loss variable $X = g(Z_1, \ldots, Z_K)$ is not different from a statistic, both are functions of random variables. The Hoeffding decomposition[1] enables us to express portfolio loss as a sum of functions of all subsets of factor risks and Euler's allocation method can then be applied to this decomposition. However, we have to consider contributions not only from single factor risks, but also from interactions of every subset of factor risks.

Consider now there are k independent factor risks $(Z_1, \ldots, Z_K)$ with finite variances, and assume that the portfolio loss $X = g(Z_1, \ldots, Z_K)$ also has finite variance. By using the Hoeffding decomposition, X can be written as a sum of uncorrelated terms involving conditional expectations of g given sets of factor risks Z in the following way:

$$X = \sum_{A \subseteq \{1,\ldots,K\}} g_A(Z_j; j \in A) \tag{1}$$

where

$$g_A(Z_j; j \in A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \, \mathbb{E}[X \mid Z_k, k \in B]. \tag{2}$$

The interpretations of these terms are nicely given in Rosen and Saunders (2010):

"*The term $g_A(Z_j; j \in A)$ gives the best hedge (in the quadratic sense) of the residual risk driven by co-movements of the factors $Z_j, j \in A$ that cannot be hedged by considering any smaller subset $B \subset A$ of the factors.*"

2.2. The Taylor expansion

We will consider an alternative to the Hoeffding decomposition to linearize the loss model in order to be able to apply allocation

[1] Rosen and Saunders (2010) measured factor contributions to credit risk of a portfolio by using the Hoeffding decomposition of the portfolio loss.