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# Quantile hedging on equity-linked life insurance contracts with transaction costs

Alexander Melnikov*, Shuo Tong

*Department of Mathematical and Statistical Science, University of Alberta, Edmonton, Alberta, Canada, T6G2G1*

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ABSTRACT

This paper analyzes the application of quantile hedging on equity-linked life insurance contracts in the presence of transaction costs. Following the time-based replication strategy, we present the explicit expressions for the present values of expected hedging errors and transaction costs. The results are derived by using the adjusted hedging volatility $\bar{\sigma}$ proposed by Leland. Furthermore, the estimated values of expected hedging errors, transaction costs and total costs are obtained from a simulation approach for comparison. Finally, the costs of maturity guarantee for equity-linked life insurance contracts inclusive of transaction costs are discussed.



## 1. Introduction

Equity-linked life insurance products have been issued by insurance companies for decades and become increasingly popular these years. These products include equity-index annuities, variable annuities and segregated funds, etc. As a type of investment linked products, the benefit of equity-linked life insurance contracts is stochastic. It mainly depends on the performance of investment in financial market such as stocks, foreign currencies, and some insurance-type events of the contract owners, such as death or survival to a certain date. In case of some poor investment performance, equity-linked life insurance products usually come with guarantees at maturity, which make such products more attractive than the traditional ones. Hardy (2003) gave comprehensive introduction on all kinds of investment guarantees in equity-linked life insurance, by taking into account the convergence of financial and insurance market.

Hedging strategies have been commonly used to price equity-linked life insurance contracts since first papers Brennan and Schwartz (1976, 1979) and Boyle and Schwartz (1977). They applied option pricing method to replicate the payoff of the contracts. Later on, Bacinello and Ortu (1993) used the similar approach to calculate the premium. Because of the mortality risk, more and more studies pointed out that imperfect hedging strategies should be applied to deal with the pricing of these contracts. For instance, mean–variance hedging by Moeller (1998, 2001), quantile hedging by Melnikov (2004, 2006), Melnikov and Skornyakova (2005), and efficient hedging by Kirch and Melnikov (2005), Melnikov and Romanyuk (2008).

In general, one of the important assumptions in the above papers is the frictionless market without transaction costs. However, transaction costs cannot be negligible in the real world. There has been considerable amount of theoretical work devoted to option pricing with transaction costs. Leland (1985) developed a hedging strategy to approximately replicate the European call option's payoff, inclusive of transaction costs. The idea is to offset the transaction costs by using a modified volatility during hedging. The modified volatility depends on both the rate of transaction costs and the length of rebalance intervals, called the revision periods. Hodges and Neuberger (1989) designed a utility-based approach on option pricing with transaction costs. Boyle and Vorst (1992) introduced an exact replication procedure for the Cox, Ross and Rubinstein binomial model in presence of transaction costs. Bensaid et al. (1992) and Edirisinghe et al. (1993) proposed a super-replication strategy. Toft (1996) obtained the closed-form expressions for expected transaction costs, hedging errors and variance of the cash flow from a time-based hedging strategy similar to Leland (1985).

As equity-linked life insurance contracts usually have long term maturities, the insurance companies need to rebalance the hedging portfolio several times within the contract term. Inspired by the above studies related to transaction costs, it is worth investigating an appropriate hedging strategy for equity-linked life insurance contracts in presence of transaction costs. The main focus of this paper is to discuss the valuation of equity-linked life insurance contracts using quantile hedging method

* Corresponding author. Tel.: +1 780 4920568.
*E-mail addresses:* melnikov@ualberta.ca (A. Melnikov), stong2@ualberta.ca (S. Tong).

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when there are transaction costs. We consider a single premium equity-linked life insurance contract and assume the guarantee at maturity is deterministic. For simplicity, we only focus on the case that the investments of the contract are on some attractive and good performance financial risky assets, which is discussed in Remark 2.1. We first calculate the quantile hedging price for the contract without transaction costs. A hedging portfolio consisting of risk-free bonds and risky assets like stocks is held at time zero. Then, based on quantile price formula, we apply a time-dependent hedging strategy similar to Leland (1985) and Toft (1996) to rebalance the portfolio. As a result, we obtain the explicit expressions for the present value of total expected transaction costs and hedging errors.

To rebalance the portfolio, the adjusted hedging volatility $\bar{\sigma}$ (Leland's approach) is utilized, which is different from the volatility $\sigma$ of underlying risky asset. We investigate the performance of Leland's adjusted volatility $\bar{\sigma}$ in presence of transaction costs by numerical examples. In fact, there are some studies which have already examined the deviation of Leland's approach. For example, Kabanov and Safarian (1997) pointed out a flaw in Leland's main theorem convergence proof; Zhao and Ziemba (2007) numerically confirmed the findings by simulation results. They mentioned that the constraint of Leland's strategy exists only when revision period is small enough: $\Delta t \to 0$. However, this is beyond the consideration of insurance companies. From the practical point of view, insurance companies cannot adjust the positions frequently. Otherwise, it is prohibitively expensive when there are transaction costs. Hence, the critics of Leland's approach cannot be directly applied to equity-linked life insurance contract case. We think this is a specific feature and we believe that our approximate price will be useful basically for applied research and insurance practice.

The similar time-based hedging strategy is applied to the segregated fund by Boyle and Hardy (1997) and Hardy (2000). They calculated the price of the contract's maturity guarantee when there are transaction costs. The results were based on stochastic simulation, but Leland's adjusted volatility was not considered. In our study, the time-based strategy from simulation in Boyle and Hardy (1997) is also applied to obtain the estimated transaction costs and hedging errors during quantile hedging. The results are used to compare with the ones calculated from explicit expressions.

This paper is organized as follows: in Section 2, the quantile price of equity-linked life insurance contract is obtained without transaction costs. In Section 3, first, we introduce total expected hedging errors and transaction costs while deriving the explicit formulae for them. Then, the numerical results are analyzed across different contract maturities and rebalance intervals. In Section 4, we discuss the estimated total expected hedging errors, transaction costs and total hedging costs using a simulation basis. Besides, the comparison with results in Section 3 is conducted. In addition, we discuss the price of the maturity guarantee for a single premium contract in presence of transaction costs. Section 5 gives the conclusion for the paper.

## 2. Quantile hedging method and the premium for equity-linked life insurance contract

In this section, we will briefly introduce quantile hedging method and its application on equity-linked life insurance to calculate the premium for the contract without consideration of transaction costs. We assume that we have a typical Black–Scholes–Merton setting: a financial market with the bond price $B_t$ and the risky asset price $S_t$ satisfying:

$$dB_t = rB_t dt \to B_t = B_0 e^{rt} \tag{2.1}$$

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma dW_t) \to S_t \\ &= S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t\right). \end{aligned} \tag{2.2}$$

where $W_t$ is a Wiener process defined on a complete probability space $\left(\Omega, F, (F_t)_{t\in[0,T]}, P\right)$, $r$ is the constant risk-free interest rate, $\mu$ is the constant mean rate of return on the risky asset, and $\sigma$ is the constant volatility of risky asset.

The equivalent martingale measure $P^*$ is unique, and its density is given by

$$Z_t = \left.\frac{dP^*}{dP}\right|_{F_t} = \exp\left(-\theta W_t - \frac{\theta^2}{2}t\right), \quad \theta = \frac{\mu - r}{\sigma}. \tag{2.3}$$

We work on a single premium equity-linked life insurance contract. The payoff of such life insurance products occurs on or after the maturity of policy, provided some prespecified event did not happen prior to the maturity date. The insured of a single premium equity-linked life insurance contract is able to receive the payoff provided that s/he is alive at the maturity to collect that payoff. We assume the contract provides a maturity guarantee $K$. In Ekern and Persson (1996), they introduced different types of guarantees. In this paper, we consider $K$ is either fixed or deterministic, which can both be treated as non-stochastic in calculations. The payoff $H_T$ depends on the value of one unit of risky asset or the guaranteed amount $K$, whichever is greater, at maturity date $T$. Payoff $H_T$ is defined as:

$$H_T = \max(S_T, K) = S_T I\{S_T \ge K\} + KI\{S_T < K\} \tag{2.4}$$

where $I\{\cdot\}$ is the indicator function.

Let $T_x$ be a nonnegative random variable, defined on another probability space $\left(\tilde{\Omega}, \tilde{F}, \tilde{P}\right)$. This random variable represents the remaining life time of an $x$-year old policyholder. Denote ${}_tp_x = \tilde{P}(T_x > t)$ the survival probability of this policyholder. It follows from the financial and mortality risk assumptions that $T_x$ is independent of all processes reflecting financial quantities. Based on discussions in Melnikov and Skornyakova (2005), and Melnikov and Romanyuk (2008), it is not possible for the insurance company to obtain a perfect hedging strategy to hedge its payoff $H_T$ because of mortality risk. The premium $X_0$ for a single premium contract can be calculated as:

$$\begin{aligned} X_0 &= E^* \times \tilde{E}\left(H_T e^{-rT} I\{T_x > T\}\right) \\ &= E^*\left(H_T e^{-rT}\right) {}_T p_x \end{aligned} \tag{2.5}$$

where $E^*(\cdot)$ is the expectation w.r.t martingale measure $P^*$. Note that the value of survival probability ${}_Tp_x$ is between 0 and 1, we obtain the following inequality:

$$X_0 < H_0 = E^*\left(He^{-rT}\right). \tag{2.6}$$

Eq. (2.6) implies that the initial amount $X_0$ collected by the insurance company from selling the contract is strictly less than the amount $H_0$, which is needed to hedge the payoff perfectly. Facing an initial budget constraint, it is impossible for the insurance company writing the contract to achieve hedging with probability 1. In this case, quantile hedging developed by Follmer and Leukert (1999) can be applied effectively to obtain an optimal hedging subject to a budget constraint.

As shown in Follmer and Leukert (1999), the proposed quantile hedge $\pi^*$ maximizes the probability of successful hedging. $\pi^*$ is proved to be unique and it coincides with the perfect hedge for a modified contingent claim $H_T^*$. $H_T^*$ has the expression $H_T^* = H_T I\{A^*\}$, where $H_T$ is the original contingent claim, $A^*$ is the maximal set of successful hedging on which the original contingent claim $H_T$ can be hedged with maximal probability of success. Set $A^*$ has the expression $\left\{1/Z_T > a^* e^{-rT} H_T\right\}$, where $a^*$ is a constant. According to Follmer and Leukert (1999), constant $a^*$