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Joint tail of ECOMOR and LCR reinsurance treaties

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ABSTRACT

Researchers in actuarial sciences have investigated the tail behavior of the LCR and ECOMOR reinsurance treaties separately for managing extreme risks in reinsurance business. In practice, a reinsurance company may possess these two treaties simultaneously. Therefore, investigating the joint tail behavior of these two treaties is practically useful in risk management. This paper derives the asymptotic limit of the joint tail of these two reinsurance treaties under the setup of Jiang and Tang (2008).

1. Introduction

Let $\{X_i\}$ be a sequence of independent and identically distributed positive claim sizes according to a counting process $N(t)$. Let $X_1^* \le \cdots \le X_{N(t)}^*$ denote the order statistics of $X_1, \ldots, X_{N(t)}$. In order to manage huge losses in insurance business, the LCR and ECOMOR reinsurance treaties were introduced by Ammeter (1964) and Thépaut (1950), respectively, which are defined as

$$L_l(t) = \sum_{i=1}^{l} X^*_{N(t)-l+i} I(N(t) \ge l) \quad \text{and}$$
$$E_l(t) = \sum_{i=1}^{l} (X^*_{N(t)-l+i} - X^*_{N(t)-l+1}) I(N(t) \ge l), \tag{1}$$

where l is a positive integer.

These two popular treaties have received some studies in the actuarial literature with focus on two quantities: the expectation with interpretation as useful in calculating premium (see Kremer, 1985, 1998) and tail probability with interpretation as predicting extreme events (see Ladoucette and Teugels, 2006; Jiang and Tang, 2008; Asimit and Jones, 2008; Hashorva and Li, 2013, and references therein).

In practice, a reinsurance company may possess these two types of treaties simultaneously, and so it is important and useful to investigate the joint tail behavior of LCR and ECOMOR for managing catastrophic risks. Let G_L and G_E denote the distribution functions of $L_l(t)$ and $E_l(t)$, respectively, for some fixed positive integer l and $t > 0$. Motivated by the multivariate extreme value theory (see De Haan and Ferreira, 2006), the joint tail of these two treaties is defined as the asymptotic behavior of $P(G_L(L_l(t)) > 1 - sx, G_E(E_l(t)) > 1 - sy)$ for some given $x, y > 0$ as $s \to 0$. Study of such a limit plays an important role in managing extreme risks for a reinsurance company.

We organize this paper as follows. Section 2 presents the main results for the asymptotic behavior of the above joint tail and the tail behavior of a linear combination of these two treaties under the same setup in Jiang and Tang (2008). Proofs are put in Section 3.

2. Main results

Throughout we follow the setup in Jiang and Tang (2008) by assuming that the claim sizes $\{X_i\}$ are independent of the counting process $N(t)$ and the common distribution function F of $X_i's$ satisfies

$$\lim_{x\to\infty} \frac{1 - F(x+y)}{1 - F(x)} = e^{-\gamma y} \quad \text{and}$$
$$\lim_{x\to\infty} \frac{P(X_1 + X_2 > x)}{P(X_1 > x)} \in (0, \infty) \tag{2}$$

for all $y \ge 0$ and some $\gamma \ge 0$. Examples and properties of distribution functions satisfying (2) can be found in Embrechts (1983), Cline (1986) and Embrechts et al. (1997).

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Under the above setting and some other conditions, Jiang and Tang (2008) showed that

$$\lim_{x\to\infty}\frac{P(L_l(t)>x)}{1-F(x)}=\sum_{n=l}^{\infty}P(N(t)=n)\frac{n!}{(n-l+1)!(l-2)!}\times\int_0^{\infty}e^{\gamma v}\left\{\int_v^{\infty}e^{\gamma u}F(du)\right\}^{l-2}F^{n-l+1}(dv):=c_L(\gamma)\tag{3}$$

and

$$\lim_{x\to\infty}\frac{P(E_l(t)>x)}{1-F(x)}=\sum_{n=l}^{\infty}P(N(t)=n)\frac{n!}{(n-l+1)!(l-2)!}\times\int_0^{\infty}e^{-(l-1)\gamma v}\left\{\int_v^{\infty}e^{\gamma u}F(du)\right\}^{l-2}F^{n-l+1}(dv):=c_E(\gamma)\tag{4}$$

for fixed $l\ge 2$ and $t>0$. Note that $c_L(0)=c_E(0)$.

As argued in the introduction, a reinsurance company may need to understand well the joint tail behavior of $L_l(t)$ and $E_l(t)$ for managing its catastrophic risks. Like the study of multivariate extremes, we define the joint tail as

$$H(x,y)=\lim_{s\to 0}s^{-1}P(G_L(L_l(t))>1-sx,G_E(E_l(t))>1-sy)\quad\text{for } x,y>0,$$

where G_L and G_E denote the distribution functions of $L_l(t)$ and $E_l(t)$, respectively. When $H(x,y)\equiv 0$, the two variables are called asymptotic independence. Otherwise they are called asymptotic dependence. When the variables are asymptotically independent, the standard bivariate extreme value theory cannot be employed to predict some type of rare events directly, and some additional assumptions are required for extrapolating the dependence function into a far tail region. When $H(x,y)=\min(x,y)$, the two variables are completely tail dependent. To distinguish asymptotic independence and asymptotic dependence, Ledford and Tawn (1996, 1997) introduced the coefficient of tail dependence. It is well known that a bivariate normal distribution with correlation coefficient between -1 and 1 is asymptotically independent. The following theorem shows that the two treaties are asymptotically dependent and become completely tail dependent when $\gamma=0$.

Theorem 1. *Under condition (2) and $E\{N(t)\}^l<\infty$ for some fixed $l\ge 2$ and $t>0$,*

(i) *if $\gamma=0$, then for $x,y>0$*

$$\lim_{s\to 0}s^{-1}P(G_L(L_l(t))>1-sx,G_E(E_l(t))>1-sy)=\min(x,y);$$

(ii) *if $\gamma>0$ and $xc_E(\gamma)<yc_L(\gamma)$ for $x,y>0$, then*

$$\lim_{s\to 0}s^{-1}P(G_L(L_l(t))>1-sx,G_E(E_l(t))>1-ty)$$
$$=\sum_{n=l}^{\infty}P(N(t)=n)\frac{n!x}{(n-l+1)!(l-2)!c_L(\gamma)}\times\int_0^{-l^{-1}\gamma^{-1}\log\frac{xc_E(\gamma)}{yc_L(\gamma)}}\left\{\int_u^{\infty}e^{\gamma v}F(dv)\right\}^{l-2}\times e^{\gamma u}F^{n-l+1}(du)+\sum_{n=l}^{\infty}P(N(t)=n)\times\frac{n!y}{(n-l+1)!(l-2)!c_E(\gamma)}\times\int_{-l^{-1}\gamma^{-1}\log\frac{xc_E(\gamma)}{yc_L(\gamma)}}^{\infty}\left\{\int_u^{\infty}e^{\gamma v}F(dv)\right\}^{l-2}\times e^{-(l-1)\gamma u}F^{n-l+1}(du);$$

Table 1
Pareto distribution. We calculate $\hat{q}$ for $q=\{0.01,0.001,0.0001\}$, $b=\{0.2,0.8\}$, and $l=\{2,5\}$.

$(l,b)=$	(2, 0.2)	(5, 0.2)	(2, 0.8)	(5, 0.8)
$q=0.01$	0.00958	0.01050	0.00950	0.01063
$q=0.001$	0.00081	0.00081	0.00090	0.00099
$q=0.0001$	0.00010	0.00012	0.00009	0.00009

(iii) *if $\gamma>0$ and $xc_E(\gamma)\ge yc_L(\gamma)$ for $x,y>0$, then*

$$\lim_{s\to 0}s^{-1}P(G_L(L_l(t))>1-sx,G_E(E_l(t))>1-sy)=y.$$

Remark 1. It follows from Theorem 1 that the coefficient of tail dependence of $L_l(t)$ and $E_l(t)$ is $\lim_{s\to 0}s^{-1}P(G_L(L_l(t))>1-s,G_E(E_l(t))>1-s)=1$ when either $\gamma=0$ or $\gamma>0$ and $c_E(\gamma)\ge c_L(\gamma)$. Although the coefficient of tail dependence indicates strongest tail dependence for these two cases, Theorem 1(i) says that the tail of the two reinsurance treaties is completely dependent only when $\gamma=0$. In this case, it is not necessary to hold both treaties when extreme risks are concerned. Hence, it is useful to test $H_0:\gamma=0$ against $H_a:\gamma>0$; see Fraga Alves et al. (2009), where a slightly different assumption from (2) is imposed.

Remark 2. It remains interesting to investigate the tail dependence for $L_{l_1}(t)$ and $E_{l_2}(t)$ with different l_1 and l_2 and under some different distribution assumption for the claim size such as gamma-like tails in Hashorva and Li (2013).

Another interesting question is to study the tail behavior of a linear combination $aL_l(t)+bE_l(t)$ for some $a,b\ge 0$ and $a+b>0$, which is given in the following theorem.

Theorem 2. *Under conditions of Theorem 1, we have*

$$\lim_{x\to\infty}\frac{P(aL_l(t)+bE_l(t)>x)}{1-F\left(\frac{x}{a+b}\right)}=\sum_{n=l}^{\infty}P(N(t)=n)\frac{n!}{(n-l+1)!(l-2)!}\times\int_0^{\infty}e^{\frac{a-b(l-1)}{a+b}\gamma u}\left\{\int_u^{\infty}e^{\gamma v}F(dv)\right\}^{l-2}F^{n-l+1}(du)$$

for any $a,b\ge 0$ and $a+b>0$.

Remark 3. Again, it is of interest to extend the result in Theorem 2 to a linear combination $aL_{l_1}(t)+bE_{l_2}(t)$ with different l_1 and l_2.

Next we examine the approximation accuracy in Theorem 2 by considering Pareto distribution $F_P(x)=1-x^{-1}$ for $x\ge 1$ and Inverse Gaussian distribution with density $f_{IG}(x;\lambda)=(\frac{\lambda}{2\pi x^3})^{1/2}\exp\{-\frac{\lambda(x-1)^2}{2x}\}$ for $x>0$. Note that (2) holds with $\gamma=0$ for the Pareto distribution and $\gamma=\lambda/2$ for the above Inverse Gaussian distribution. We take $t=1$, $a+b=1$, $l=2$ or 5, and let $N(t)$ be a Poisson process with intensity 10. First we find $x=x_q$ for $q=\{0.01,0.001,0.0001\}$ such that

$$\{1-F(x)\}\sum_{n=l}^{\infty}P(N(t)=n)\frac{n!}{(n-l+1)!(l-2)!}\times\int_0^{\infty}e^{(1-bl)\gamma u}\left\{\int_u^{\infty}e^{\gamma v}F(dv)\right\}^{l-2}F^{n-l+1}(du)=q,$$

and then we simulate 100, 000 random samples from each of the above settings to obtain the empirical quantile $\hat{q}$ for estimating $P(aL_l(t)+bE_l(t)>x_q)$. The values of $\hat{q}$ are reported in Tables 1 and 2 below, which show that the approximation for $\gamma=0$ is more