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Second order risk aggregation with the Bernstein copula

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ABSTRACT

We analyze the tail of the sum of two random variables when the dependence structure is driven by the Bernstein family of copulas. We consider exponential and Pareto distributions as marginals. We show that the first term in the asymptotic behavior of the sum is not driven by the dependence structure when a Pareto random variable is involved. Consequences on the Value-at-Risk are derived and examples are discussed.

1. Introduction

Risk aggregation is now a classical topic for researchers and practitioners both in Finance and Insurance. The regulatory frameworks of Basel (I, II and III) and Solvency (I and II) make it crucial to understand how the various risks within a portfolio combine in order to estimate future losses. The naive way to proceed is to add up individual risks. This is of course not the best solution since assets are usually correlated, especially in the midst of financial crises.

A very popular approach seeks to analyze the marginal behavior of the risks separately from their dependence structure. The natural way to proceed is to resort to copulas. More precisely, if $F_1, \dots, F_n$ are the cumulative distribution functions (c.d.f.) of the absolutely continuous random variables (r.v.) $X_1, \dots, X_n$, then the copula related to these r.v.s is the unique mapping C such that

$$P[X_1 \le x_1, \dots, X_n \le x_n] = C(F_1(x_1), \dots, F_n(x_n)), \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

We refer to the monograph (Nelsen, 2006) for more details on the subject. Once the marginals and the copula are specified, it becomes possible, though often complicated, to study the behavior of $P[X_1 + \dots + X_n > x]$, as $x \to \infty$. It seems intuitive that this behavior should match, in some sense, that of the random variable which has the heaviest tail. In fact, this is very often true, especially when heavy tails are involved.

More precisely, when X_1 and X_2 have the same heavy-tailed distribution, then except for a handful of cases (see for instance Theorem 2.10 in Albrecher et al., 2006),

$$\lim_{x\to\infty} \frac{P[X_1 + X_2 > x]}{P[X_1 > x]} = c > 0. \quad (1)$$

When X_1 and X_2 have non identical heavy tailed distribution, then these results usually hold if X_1 is the random variable with the heaviest tail. Many results of this form have blossomed and we mention some of them in the footnote.[1]

Note that the asymptotic result in (1) can be very imprecise, especially for the computation of extreme quantiles. For instance, consider the random variable with c.d.f. given by

$$F(x) = P[X \le x] = \left(1 - \frac{1}{2x^2} - \frac{1}{2x^3}\right) \mathbf{1}_{\{x \ge 1\}},$$

then obviously, $2P[X > x] = x^{-2} + x^{-3}$ when $x \to \infty$. If we consider the quantile at the level $p = 95\%$, then $F^{-1}(0.95) \approx 3.577$. However, if we consider the first order approximation $2P[X > x] \sim x^{-2}$, then the same quantile is estimated at 3.162, which is a 13% relative error.

Second order expansions have emerged recently in the copula related literature (Kortschak, 2012; Kortschak and Hashorva, 2013)

[1] See Albrecher et al. (2006), Alink et al. (2004), Assmussen and Rojas-Nandayapa (2008), Barbe et al. (2006), Davis and Resnick (1996), Embrechts et al. (2009), Foss and Richard (2010), Geluk and Tang (2009), Goovaerts et al. (2005), Ko and Tang (2008), Kortschak and Albrecher (2009), Kortschak (2012), Laeven et al. (2005), Mitra and Resnick (2009) and Wüthrich (2003).

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and provide, by definition, better estimates for extreme quantiles. The purpose of this article is to show that the Bernstein family of copulas enables to get second order asymptotics for the sum of two dependent variables when the univariate distributions are exponential and/or Pareto. In fact, as is shown in the proofs, higher orders can easily be derived.

The remainder of the paper is structured as follows. In Section 2, we present the setting and our main results. In Section 3, we illustrate our findings with a particular example. An extension of our method is provided in Section 4 and Section 5 concludes. All of the proofs are provided in the Appendix.

2. Notations and main results

We first introduce the two types of parametric families of distributions which we will consider. We denote exponentially distributed r.v.s by the letter X, that is, for $\lambda_i > 0$, $X_i^{(\lambda_i)} \overset{d}{=} \mathcal{E}(\lambda_i)$ will have density $f_i(x) = \lambda_i e^{-\lambda_i x}\mathbf{1}_{\{x\geq 0\}}$ and c.d.f. $F_i(x) = (1 - e^{-\lambda_i x})\mathbf{1}_{\{x\geq 0\}}$.

Likewise, for $a_j > 0$ and $b_j > 1$, $Y_j^{(a_j,b_j)} \overset{d}{=} \mathcal{P}(a_j, b_j)$ will follow a Pareto distribution with density and c.d.f. given by

$$g_j(x) = b_j a_j^{b_j}(a_j + x)^{-(b_j+1)}\mathbf{1}_{\{x\geq 0\}}, \qquad (1 - a_j^{b_j}(a_j + x)^{-b_j})\mathbf{1}_{\{x\geq 0\}}.$$

We impose $b_j > 1$ so that the variables we consider have finite mean. In many contexts, these random variables can be used to model losses (Finance, Insurance, Risk Management, etc.). Note that a scaling and a shift $x \mapsto (x-\mu)/\sigma$ in the c.d.f. give the law of $\sigma X+\mu$ and hence the model becomes more flexible effortlessly. For notational convenience, we shall however henceforth set $\mu = 0$ and $\sigma = 1$.

Now that the marginals have been defined, we turn to the dependence structure. We start with the definition of the bivariate Bernstein copula, which is a particular case of the copula introduced in Sancetta and Satchell (2004):

$$C^B(u, v) = \sum_{i=0}^{m}\sum_{j=0}^{m} a\left(\frac{i}{m}, \frac{j}{m}\right)\binom{m}{i}\binom{m}{j} u^i(1-u)^{m-i}v^j(1-v)^{m-j}. \quad (2)$$

Sancetta and Satchel have shown that C^B is indeed a copula whenever for $v_i = 0, 1, \ldots, m-1$, the function a verifies

$$a\left(\frac{v_1}{m}, \frac{v_2}{m}\right) - a\left(\frac{v_1+1}{m}, \frac{v_2}{m}\right) - a\left(\frac{v_1}{m}, \frac{v_2+1}{m}\right) + a\left(\frac{v_1+1}{m}, \frac{v_2+1}{m}\right) \geq 0$$

and

$$\max\left(0, \frac{v_1}{m} + \frac{v_2}{m} - 1\right) \leq a\left(\frac{v_1}{m}, \frac{v_2}{m}\right) \leq \min\left(\frac{v_1}{m}, \frac{v_2}{m}\right).$$

Note that this last condition implies $a(0, v_2) = a(v_1, 0) = 0$.

For notational convenience, we will use the simpler form

$$C^B(u, v) = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}u^i v^j, \quad (u, v) \in [0, 1]^2, \quad (3)$$

where

$$c_{i,j} = \sum_{k=1}^{i}\sum_{l=1}^{j} a\left(\frac{k}{m}, \frac{l}{m}\right)\binom{m}{k}\binom{m}{l}\binom{m-k}{i-k}\binom{m-l}{j-l} \times (-1)^{i-k+j-l}.$$

It is obvious that $c_{i,j} = c_{j,i}$ and moreover, the equality $C^B(1, 1) = 1$ yields

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j} = 1.$$

More precisely, using first order partial derivatives and conditional probabilities, we get

$$\sum_{1\leq i,j\leq m} c_{i,j} i u^{i-1} = \sum_{1\leq i,j\leq m} c_{i,j} j v^{j-1} = 1, \quad \forall u, v \in (0, 1),$$

which translates into

$$\sum_{j=1}^{m} c_{1,j} = \sum_{i=1}^{m} c_{i,1} = 1,$$

$$\sum_{j=1}^{m} c_{k,j} = \sum_{i=1}^{m} c_{i,k} = 0, \quad k \in \{2, \ldots, m\}.$$

This implies that for any function f well-defined on integers,

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}f(i) = \sum_{i=1}^{m} f(i)\sum_{j=1}^{m} c_{i,j} = \sum_{j=1}^{m} f(j)\sum_{i=1}^{m} c_{i,j} = f(1). \quad (4)$$

Originally, Bernstein copulas were introduced with a view to approximate other copulas. However, they have since then been used in various contexts (financial and actuarial notably), as is pointed out in the introduction of Tavin (2013).

As the following theorems show, the simple form (3) enables tractable computations for the asymptotics of the survival function of the sum of two dependent variables.

Theorem 2.1 (*Exponential Distributions*)**.** *Let (X_1, X_2) follow a bivariate distribution with exponential marginals and the Bernstein copula as dependence structure, then as $z \to \infty$,*

$$P\left[X_1^{(\lambda)} + X_2^{(\lambda)} > z\right] = \lambda z \left(\sum_{\substack{1\leq i\leq m\\ 0\leq j\leq m}} c_{i,j} ij\right) e^{-\lambda z} + C_e e^{-\lambda z} + o(e^{-\lambda z})$$

and if $\lambda_1 < \lambda_2$ and $\lambda_1/\lambda_2 \notin \{1/2, 2/3, 1/3\}$, then

$$P\left[X_1^{(\lambda_1)} + X_2^{(\lambda_2)} > z\right] = C_1 e^{-\lambda_1 z} + C_2 e^{-\lambda_2 z} + C_3 e^{-2\lambda_1 z} + o(e^{-2\lambda_1 z}),$$

where the constants C_e, C_1, C_2 and C_3 are provided in the proof in the Appendix.

Theorem 2.2 (*Pareto Distributions*)**.** *Let (Y_1, Y_2) follow a bivariate distribution with Pareto marginals (with $b_2 > b_1 > 1$) and the Bernstein copula as dependence structure, then as $z \to \infty$,*

$$P\left[Y_1^{(a_1,b_1)} + Y_2^{(a_2,b_2)} > z\right] = a_1^{b_1} z^{-b_1} + a_2^{b_2} z^{-b_2} + C_p z^{-b_1-1} + O(z^{-b_2-1} + z^{-b_1-b_2} + z^{-b_1-2}),$$

where the constant C_p is provided in the Appendix.

Theorem 2.3 (*Exponential and Pareto Distributions*)**.** *Let (X, Y) follow a bivariate distribution with X following an exponential law and Y following a Pareto law and the Bernstein copula as dependence structure, then as $z \to \infty$,*

$$P[X^{(\lambda)} + Y^{(a,b)} > z] = a^b z^{-b} + C_{pe} z^{-b-1} + o(z^{-b-1})$$

where the constant C_{pe} is provided in the proof in the Appendix.