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Explicit solutions of optimal consumption, investment and insurance problems with regime switching

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HIGHLIGHTS

- We obtain explicit solutions for simultaneous optimal consumption, investment and insurance problems.
- The solution depends strongly on the regime of the economy.
- In our model, optimal insurance is either no insurance or deductible insurance.
- We determine the conditions under which it is optimal to buy insurance in our model.
- We calculate the advantage of buying insurance in our model.

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ABSTRACT

We consider an investor who wants to select his optimal consumption, investment and insurance policies. Motivated by new insurance products, we allow not only the financial market but also the insurable loss to depend on the regime of the economy. The objective of the investor is to maximize his expected total discounted utility of consumption over an infinite time horizon. For the case of hyperbolic absolute risk aversion (HARA) utility functions, we obtain the first explicit solutions for simultaneous optimal consumption, investment, and insurance problems when there is regime switching. We determine that the optimal insurance contract is either no-insurance or deductible insurance, and calculate when it is optimal to buy insurance. The optimal policy depends strongly on the regime of the economy. Through an economic analysis, we calculate the advantage of buying insurance.

1. Introduction

In the classical consumption and investment problem, a risk-averse investor wants to maximize his expected discounted utility of consumption by selecting optimal consumption and investment strategies. Merton (1969) was the first to obtain explicit solutions to this problem in continuous time. Many generalizations to Merton's work can be found in Karatzas (1996), Karatzas and Shreve (1998), Sethi (1997), et cétera. In the traditional models for consumption and investment problems, there is only one source of risk that comes from the uncertainty of the stock prices. But in real life, apart from the risk exposure in the financial market, investors often face other random risks, such as property–liability risk and credit default risk. Thus, it is more realistic and practical to extend the traditional models by incorporating an insurable risk. When an investor is subject to an additional insurable risk, buying insurance is a trade-off decision. On one hand, insurance can provide the investor with compensation and then offset capital losses if the specified risk events occur. On the other hand, the cost of insurance diminishes the investor's ability to consume and therefore reduces the investor's expected utility of consumption.

The initial optimal insurance problem studies an individual who is subject to an insurable risk and seeks the optimal amount of insurance under the utility maximization criterion. Using the expected value principle for premium, Arrow (1963) found that the

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optimal insurance is deductible insurance in discrete time. Promislow and Young (2005) reviewed optimal insurance problems (without investment and consumption). They proposed a general market model and obtained explicit solutions to optimal insurance problems under different premium principles, such as variance principle, equivalent utility principle, Wang's principle, et cétera.

Moore and Young (2006) combined Merton's optimal consumption and investment problem and Arrow's optimal insurance problem in continuous time. They found explicit or numerical solutions for different utility functions (although they did not verify rigorously that the obtained strategies were indeed optimal). Perera (2010) revisited Moore and Young's work by considering their problem in a more general Levy market, and applied the martingale approach to obtain explicit optimal strategies for exponential utility functions.

In traditional financial modeling, the market parameters are assumed to be independent of general macroeconomic conditions. However, historical data and empirical research show that the market behavior is affected by long-term economic factors, which may change dramatically as time evolves. Regime switching models use a continuous-time Markov chain with a finite-state space to represent the uncertainty of those long-term economic factors.

Hamilton (1989) introduced a regime switching model to capture the movements of the stock prices and showed that the regime switching model represents the stock returns better than the model with deterministic coefficients. Thereafter, regime switching has been applied to model many financial and economic problems (see for instance Sotomayor and Cadenillas, 2009, for some references).

In the insurance market, insurance policies can depend on the regime of the economy. In the case of traditional insurance, the underwriting cycle has been well documented in the literature. Indeed, empirical research provides evidence for the dependence of insurance policies' underwriting performance on external economic conditions (see for instance Grace and Hotchkiss, 1995; Haley, 1993 on property–liability insurance; and Chung and Weiss, 2004 on reinsurance). In the case of non-traditional insurance, by investigating the comovements of credit default swap (CDS) and the bond/stock markets, Norden and Weber (2007) found that CDS spreads are negatively correlated with the price movements of the underlying stocks and such cointegration is affected by the corporate bond volume.

In this paper, we use an observable continuous-time finite-state Markov chain to model the regime of the economy and allow both the financial market and the insurance market to depend on the regime. Our objective is to obtain simultaneously optimal consumption, investment and insurance policies for a risk-averse investor who wants to maximize his expected total discounted utility of consumption over an infinite time horizon. We extend Sotomayor and Cadenillas (2009) by including a random loss in the model and an insurance policy in the control. The most important difference between the model of Moore and Young (2006) and our paper is that they do not allow regime switching, while we allow regime switching in both the financial market and the insurance market.

2. The model

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in which a standard Brownian motion W and an observable continuous-time, stationary, finite-state Markov chain ϵ are defined. Denote by $\mathcal{S} = \{1, 2, \dots, S\}$ the state space of this Markov chain, where S is the number of regimes in the economy. The matrix $Q = (q_{ij})_{S\times S}$ denotes the strongly irreducible generator of ϵ, where $\forall\, i \in \mathcal{S}$, $\sum_{j\in\mathcal{S}} q_{ij} = 0$, $q_{ij} > 0$ when $j \neq i$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$.

We consider a financial market consisting of two assets, a bond with price P_0 (riskless asset) and a stock with price P_1(risky asset), respectively. Their price processes are driven by the following dynamics:

$$dP_0(t) = r_{\epsilon(t)} P_0(t)dt,$$

$$dP_1(t) = P_1(t)(\mu_{\epsilon(t)}dt + \sigma_{\epsilon(t)}dW(t)),$$

with initial conditions $P_0(0) = 1$ and $P_1(0) > 0$. The coefficients r_i, μ_i and σ_i, $i \in \mathcal{S}$, are all positive constants.

An investor chooses $\pi = \{\pi(t), t \geq 0\}$, the proportion of wealth invested in the stock, and a consumption rate process $c = \{c(t), t \geq 0\}$. We assume that the investor is subject to an insurable loss $L(t, \epsilon(t), X(t))$, where $X(t)$ denotes the investor's wealth at time t. We shall use the short notation L_t to replace $L(t, \epsilon(t), X(t))$ if there is no confusion. We use a Poisson process N with intensity $\lambda_{\epsilon(t)}$, where $\lambda_i > 0$ for every $i \in \mathcal{S}$, to model the occurrence of this insurable loss. In the insurance market, there are insurance policies available to insure against the loss L_t. We further assume that the investor can control the payout amount $I(t)$, where $I(t) : [0, \infty) \times \Omega \mapsto [0, \infty)$ and $I(t, \omega) := I_t(L(t, \epsilon(t, \omega), X(t, \omega)))$, or in short, $I(t) = I_t(L_t)$. For example, if $\Delta N(t_0) = 1$, then at time t_0 the investor suffers a loss of amount L_{t_0} but receives a compensation of amount $I_{t_0}(L_{t_0})$ from the insurance policy, so the investor's net loss is $L_{t_0} - I_{t_0}(L_{t_0})$. Following the premium setting used in Moore and Young (2006) (the famous expected value principle), we assume investors pay premium continuously at the rate P given by

$$P(t) = \lambda_{\epsilon(t)}(1 + \theta_{\epsilon(t)})E[I_t(L_t)],$$

where the positive constant θ_i, $i \in \mathcal{S}$, is known as the loading factor in the insurance industry. Such extra positive loading comes from insurance companies' administrative cost, tax, profit, et cétera.

Following Sotomayor and Cadenillas (2009), we assume that the Brownian motion W, the Poisson process N and the Markov chain ϵ are mutually independent. We also assume that the loss process L is independent of N. We take the $\mathbb{P}$-augmented filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by W, N, L and ϵ as our filtration and define $\mathcal{F} := \sigma(\cup_{t\geq 0} \mathcal{F}_t)$. An investor with triplet strategies $u(t) := (\pi(t), c(t), I(t))$ has a wealth process X given by

$$\begin{aligned} dX(t) = \big(&r_{\epsilon(t)}X(t) + (\mu_{\epsilon(t)} - r_{\epsilon(t)})\pi(t)X(t) - c(t) \\ &- \lambda_{\epsilon(t)}(1 + \theta_{\epsilon(t)}) \cdot E[I_t(L_t)]\big)dt \\ &+ \sigma_{\epsilon(t)}\pi(t)X(t)dW(t) - (L_t - I_t(L_t))\, dN(t), \end{aligned} \quad (1)$$

with initial conditions $X(0) = x > 0$ and $\epsilon(0) = i \in \mathcal{S}$.

We define the criterion function J as

$$J(x, i; u) := E_{x,i}\left[\int_0^{+\infty} e^{-\delta t}U(c(t), \epsilon(t))dt\right], \quad (2)$$

where $\delta > 0$ is the discount rate and $E_{x,i}$ means conditional expectation given $X(0) = x$ and $\epsilon(0) = i$. We assume that for every $i \in \mathcal{S}$, the utility function $U(\cdot, i)$ is $C^2(0, +\infty)$, strictly increasing and concave, and satisfies the linear growth condition

$$\exists K > 0 \text{ such that } U(y, i) \leq K(1 + y), \quad \forall\, y > 0, i \in \mathcal{S}.$$

Besides, we use the notation $U(0, i) := \lim_{y\to 0^+} U(y, i)$, $\forall\, i \in \mathcal{S}$.

We define the bankruptcy time as

$$\Theta := \inf\{t \geq 0 : X(t) \leq 0\}.$$

Since an investor consumes only when his wealth is strictly positive, we define

$$R(\Theta) := \int_\Theta^\infty e^{-\delta t}U(c(t), \epsilon(t))dt = \int_\Theta^\infty e^{-\delta t}U(0, \epsilon(t))dt.$$

A control $u := (\pi, c, I)$ is called admissible if $\{u_t\}_{t\geq 0}$ is predictable with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies $\forall\, t \geq 0$

$$E_{x,i}\left[\int_0^t c(s)ds\right] < +\infty, \qquad E_{x,i}\left[\int_0^t \sigma^2_{\epsilon(s)}\pi^2(s)ds\right] < +\infty,$$

$$E_{x,i}\left[\int_0^\Theta e^{-\delta s}U^+(c(s), \epsilon(s))ds\right] < +\infty,$$