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On the analysis of time dependent claims in a class of birth process claim count models

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ABSTRACT

An integral representation is derived for the sum of all claims over a finite interval when the claim value depends upon its incurral time. These time dependent claims, which generalize the usual compound model for aggregate claims, have insurance applications involving models for inflation and payment delays. The number of claims process is assumed to be a (possibly delayed) nonhomogeneous birth process, which includes the Poisson process, contagion models, and the mixed Poisson process, as special cases. Known simplified compound representations in these special cases are easily generalized to the conditional case, given the number of claims at the beginning of the interval. Applications to the case involving "two stages" are also considered.

1. Introduction and background

There has been much work done on the analysis of the aggregate claim distribution when claim sizes are time dependent, so that the distribution depends on their incurral time. Models involving claims inflation and delays in claim payments may be formulated in this manner.

The assumptions as to the number of claims incurred process $\{N_t, t \geq 0\}$ are of utmost importance in terms of the analysis. In particular, under the Poisson and mixed Poisson process assumptions, simplified representations in terms of compound distributions for the sum of all time dependent claims for claims incurred in $(0, t)$ are available (e.g., Guo et al. (2013) and references therein). On the other hand, the analysis when $\{N_t, t \geq 0\}$ is a (possibly delayed) renewal (Sparre Andersen) process and related processes is much more challenging (see, e.g., Léveillé and Adekambi (2012), Woo and Cheung (2013), as well as references therein).

In this paper, we consider the case when $\{N_t, t \geq 0\}$ is a nonhomogeneous birth process, a model which is shown to be particularly suited for use in a time dependent claims context. The reader is referred to Klugman et al. (2013, Chapter 7) and references therein for a description of this process. In Section 2, we derive an integral representation for the sum of the claim values over the interval (s, t) given that $N_s = k$. This representation is convenient for this situation due to the direct incorporation of the claim occurrence times into the analysis, and is no more complex than the unconditional sum over $(0, t)$. Important special cases of the nonhomogeneous birth process are the (nonhomogeneous) Poisson process (e.g. Ross (1996)), the so-called contagion models (e.g. Bühlmann (1970, Chapter 2)), and the mixed Poisson process (e.g., Grandell (1997, Chapter 4)). In these special cases, simplified compound distribution representations for the conditional distributions over (s, t) are obtained in Section 3 from the general result, and these often generalize known results. Also, an application in the context of a "two-stage" nonhomogeneous birth process is briefly discussed in Section 4.

2. The general setup

Nonhomogeneous birth processes are discussed in detail by Bühlmann (1970), Grandell (1997), Klugman et al. (2013), and references therein. Only a few details relevant for the present analysis are presented here. Of central importance to the analysis are the transition probabilities

$$p_{k,k+n}(s,t) = \Pr(N_t - N_s = n | N_s = k), \quad n = 0, 1, 2, \ldots, \tag{1}$$

with probability generating function (pgf)

$$P_{k,s,t}(z) = \sum_{n=0}^{\infty} p_{k,k+n}(s,t) z^n. \tag{2}$$

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The marginal probabilities are given (under the assumption that $N_0 = 0$) by $p_n(t) = \Pr(N_t = n) = p_{0,n}(0,t)$. The transition probabilities are characterized by the so-called *transition intensities* $\{\lambda_m(t), m = 0, 1, 2, \ldots\}$. It is known that

$$p_{m,m}(s,t) = e^{-\int_s^t \lambda_m(y)dy}, \quad m = 0, 1, 2, \ldots, \tag{3}$$

and for general $n \geq 1$, the probabilities given by (1) may be obtained recursively in n. Explicit formulas are obtainable for some choices of $\lambda_m(t)$. See Klugman et al. (2013, Chapter 7), Willmot (2010), and references therein for more details.

In the remainder of this paper, we are interested in the behavior of the process after a fixed time s, given the value of N_s, say k. Thus, the results hold for any Markov process which behaves as a nonhomogeneous birth process thereafter. Thus for the ordinary nonhomogeneous birth process itself, no assumptions need to be made about $\lambda_m(t)$ for $m < k$, as follows from (3).

In what follows, let T_m $(m = 0, 1, 2, \ldots)$ represent the time of the m-th claim, with realization denoted by t_m. Also, let $h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k, s)$ denote the density function associated with the event that there are exactly n claims in (s,t) at times $t_{k+1} < t_{k+2} < \cdots < t_{k+n}$ where $s < t_{k+1}$ and $t_{k+n} < t$, given that $N_s = k$. This density is of central importance in what follows, and is now given explicitly.

Lemma 1. *For* $n = 1, 2, \ldots,$

$$h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k, s) = e^{-\int_s^t \lambda_{k+n}(y)dy} \prod_{m=1}^{n} \phi_{k+m}(t_{k+m}|s), \tag{4}$$

where

$$\phi_j(x|s) = \lambda_{j-1}(x)\, e^{\int_s^x \{\lambda_j(y) - \lambda_{j-1}(y)\}dy}. \tag{5}$$

Proof. It is clear from (3) that for

$$m = 1, 2, \ldots, \exp\left\{-\int_{t_{k+m-1}}^{t_*} \lambda_{k+m-1}(y)\, dy\right\}$$

may be interpreted as the probability that T_{k+m} exceeds t_*, given that $N_{t_{k+m-1}} = k+m-1$. Thus, $\lambda_{k+m-1}(y)$ is the associated failure rate, and (assuming for the moment that $t_k = s$), the joint density of $T_{k+1}, T_{k+2}, \ldots, T_{k+n}|N_s = k$ may thus be expressed as

$$\prod_{m=1}^{n} \lambda_{k+m-1}(t_{k+m})\, e^{-\int_{t_{k+m-1}}^{t_{k+m}} \lambda_{k+m-1}(y)dy}.$$

In order to have exactly n claims in (s,t), there can be no more claims in (t_{k+n}, t), with probability $\exp\left\{-\int_{t_{k+n}}^t \lambda_{k+n}(y)\, dy\right\}$, implying that

$$h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k, s) = e^{-\int_{t_{k+n}}^t \lambda_{k+n}(y)dy} \prod_{m=1}^{n} \lambda_{k+m-1}(t_{k+m})\, e^{-\int_{t_{k+m-1}}^{t_{k+m}} \lambda_{k+m-1}(y)dy}. \tag{6}$$

Simple rearrangements of (6) result in (4). □

An explicit expression for (1) follows immediately from Lemma 1.

Lemma 2. *The transition probabilities* (1) *may be expressed as*

$$p_{k,k+1}(s,t) = \int_s^t h_{1,t}(t_{k+1}|k, s)\, dt_{k+1}, \tag{7}$$

and for $n = 2, 3, \ldots,$

$$p_{k,k+n}(s,t) = \int_s^t \int_s^{t_{k+n}} \cdots \int_s^{t_{k+2}} h_{n,t}(t_{k+1}, \ldots, t_{k+n}|k, s) \times dt_{k+1} dt_{k+2} \ldots dt_{k+n}. \tag{8}$$

Proof. Integrating over all possible values of $t_{k+1}, t_{k+2}, \ldots, t_{k+n}$ results in (7) and (8). □

We now turn to the problem of interest, namely the analysis of time dependent claims. To this end, let $X_{s,t}$ be the sum total of all *claim values* for claims incurred in (s,t), where the definition of a *claim value* depends on the particular quantity of interest to be analyzed, as is discussed in more detail in the next paragraph. See Klugman et al. (2013, Section 9.1) for a discussion of this issue. We denote the conditional Laplace transform (LT) of $X_{s,t}$, given that $N_s = k$ and there are exactly n claims in (s,t) at times $t_{k+1}, t_{k+2}, \ldots, t_{k+n}$ by $\widetilde{f}_{n,t}(z|k, s, t_{k+1}, \ldots, t_{k+n})$. If we further assume that the individual claim values are independent of all other claim values, with distribution depending on nothing more than possibly the incurral time, k, s and t, then we may write

$$\widetilde{f}_{n,t}(z|k, s, t_{k+1}, \ldots, t_{k+n}) = \prod_{m=1}^{n} \widetilde{f}_t(z|k, s, t_{k+m}). \tag{9}$$

Note that in (9), $\widetilde{f}_t(z|k, s, x)$ is the LT of the claim value associated with a claim incurral at $x \in (s,t)$. While the independence assumption is not necessary, we remark that insurance applications to date involving inflation and payment delays have typically assumed that a factorization of the form (9) holds.

In particular, a model for inflation assumes that $\widetilde{f}_t(z|k, s, x) = \widetilde{f}\left(ze^{-\int_s^x \delta_y dy}\right)$ where δ_y is the net instantaneous rate of interest at time y, and $\widetilde{f}(z)$ is the LT of the amount of a claim incurred at time s. In this case, $X_{s,t}$ is the discounted value at time s of the sum of all claims incurred over (s,t). Similarly, a model for delays in claim payments may be obtained with the choice $\widetilde{f}_t(z|k, s, x) = B_x\left\{W_x(t-x) + \overline{W}_x(t-x)\widetilde{f}^*_{x,t}(z)\right\}$. In this case, $B_x(z)$ is the pgf of the number of claims resulting from a claim causing event at time x, $W_x(y) = 1 - \overline{W}_x(y)$ is the distribution function of the delay in claim payment for a claim incurred at x, and $\widetilde{f}^*_{x,t}(z)$ is the LT of the amount of a claim incurred at x, valued at t. Then $X_{s,t}$ is the value at time t of the sum of all claims incurred in (s,t) and unpaid by time t. See Klugman et al. (2013, Chapter 9) and references therein for further details on these models.

We are now in a position to state the general result for the aggregate claim values, conditional on $N_s = k$.

Theorem 3. *Given that* $N_s = k$*, the aggregate claim values associated with claims incurred in* (s,t) *has Laplace transform*

$$E\left[e^{-zX_{s,t}}|N_s = k\right] = p_{k,k}(s,t) + \sum_{n=1}^{\infty} p_{k,k+n}(s,t)\widetilde{f}_{n,t}(z|k, s), \tag{10}$$

where

$$\widetilde{f}_{1,t}(z|k, s) = \frac{\int_s^t h_{1,t}(t_{k+1}|k, s)\widetilde{f}_{1,t}(z|k, s, t_{k+1})\, dt_{k+1}}{\int_s^t h_{1,t}(t_{k+1}|k, s)\, dt_{k+1}}, \tag{11}$$

and for $n = 2, 3, \ldots,$ *(see Box I)*

Proof. Obviously, $X_{s,t} = 0$ if $N_t - N_s = 0$, and otherwise (10) follows directly by conditioning on $N_t - N_s = n$, and the n claim times $T_{k+1}, \ldots, T_{k+n}$, together with (7) and (8). □

Clearly, (11) and (12) imply that $\widetilde{f}_{n,t}(z|k, s, t_{k+1}, \ldots, t_{k+n})$ may be represented as a mixture, with mixing weights proportional to $h_{n,t}(t_{k+1}, \ldots, t_{k+n}|k, s)$. Also, it is useful to note that in the important special case when (9) holds, (4) implies that for any n, the integrand in (12) factors as a function of the integration variables $t_{k+1}, t_{k+2}, \ldots, t_{k+n}$.

While the representation of Theorem 3 is extremely general, a very useful simplification results if (9) holds and $\lambda_j(x)$ for $j = k, k+1, \ldots$ is such that $\phi_{k+m}(x|s)$, defined in (5) for $m = 1, 2, \ldots,$ factors (for fixed k and s) as a function of m multiplied by a function of x. This is the case for (possibly) nonhomogeneous versions of