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Optimal capital allocation based on the Tail Mean–Variance model

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HIGHLIGHTS

- We study capital allocation based on a novel Tail Mean–Variance model.
- General formulas for the optimal capital allocations are derived.
- Explicit formulas for optimal capital allocations are derived for multivariate elliptical distributions.
- Asymptotic allocation formulas for multivariate regular variation variables are given.

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ABSTRACT

This paper studies capital allocation problems with the aggregate risk exceeding a certain threshold. We propose a novel capital allocation rule based on the Tail Mean–Variance principle. General formulas for the optimal capital allocations are proposed. Explicit formulas for optimal capital allocations are derived for multivariate elliptical distributions. Moreover, we give asymptotic allocation formulas for multivariate regular variation variables. Various numerical examples are given to illustrate the results, and real insurance data is discussed as well.

1. Introduction and motivation

For an insurance company, the allocation of total capital to its various business units is an extremely important task. There has been growing interest in studying the optimal capital allocations. One may refer to Myers and Read (2001), Laeven and Goovaerts (2004), Frostig et al. (2007), Furman and Zitikis (2008), Tsanakas (2009), Dhaene et al. (2012), and Xu and Hu (2012) and references therein for recent developments. Recently, much more attention has been paid to the capital allocation, which considers tail risks. The two well-known risk measurements, Value-at-Risk (VaR) and tail conditional expectation (TCE), have been used as criteria for this purpose. Panjer (2002) considered the allocation rule using TCE under the multivariate normal assumption. This model was extended to multivariate elliptical distributions in Landsman and Valdez (2003), multivariate gamma and Tweedie distributions in Furman and Landsman (2005, 2006), and multivariate Pareto distributions in Chiragiev and Landsman (2007); see also Cai and Li (2005) for multivariate phase-type distributions.

Let us assume that a firm has a portfolio of risks $X_1, \ldots, X_n$. Assume that a company wishes to allocate the total capital $p = p_1 + \cdots + p_n$ to the corresponding risks. Dhaene et al. (2012) proposed a criterion, which is to set the capital amount p_i as close as possible to X_i as measured by some appropriate distance measure. More specifically, they proposed the following optimization problem to model the capital allocation problem:

$$\min_{\boldsymbol{p}\in A} \sum_{i=1}^{n} v_i \mathbb{E}\left[\zeta_i D\left(\frac{X_i - p_i}{v_i}\right)\right] \tag{1.1}$$

for

$$\boldsymbol{p} \in A = \{\boldsymbol{p} \in \Re^n : p_1 + \cdots + p_n = p\},$$

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where v_i are non-negative real numbers such that $\sum_{i=1}^n v_i = 1$, and the ζ_i are non-negative random variables such that $\mathbb{E}[\zeta_i] = 1$, and D is some suitable distance measurement function which measures the loss of allocations. This framework is quite general and includes many well-known capital allocation rules as special cases, such as Haircut, Quantile and Covariance. In fact, the idea of minimizing the loss function has been discussed in the framework of premium calculation. For example, Laeven and Goovaerts (2004) used $D(x) = \max\{x, 0\}$ as a distance measure, and Zaks et al. (2006) used a quadratic distance measure $D(x) = x^2$. This topic was further pursued in Frostig et al. (2007), where they used the general convex distance measure. This idea was generalized in Xu and Hu (2012), where they defined the following loss function:

$$L(\boldsymbol{p}) = \sum_{i=1}^{n} D(X_i - p_i).$$

They proposed the following optimization problem:

$$\min_{\boldsymbol{p}\in A} P(L(\boldsymbol{p}) \geq t), \quad \forall t \geq 0.$$

Although the allocation rule based on minimizing the loss function has brilliant advantages, there are two important issues, which have not been taken into account.

(a) Variability. Since we only rely on the magnitude of the loss function, the important factor of variability of loss function has not been incorporated into the allocation rule. The relevant idea of considering the variability has appeared in the premium calculation, see, for example, Valdez (2005) and Furman and Landsman (2006). More recently, Ostaszewski and Xu (2012) proposed a Mean–Variance framework to overcome this limitation. More specifically, they proposed the following allocation rule:

$$\begin{cases} \min_{\boldsymbol{p}\in A} \left\{ \mathbb{E}\left[\sum_{i=1}^{n} (X_i - p_i)^2\right] + \beta \mathrm{Var}\left(\sum_{i=1}^{n} (X_i - p_i)^2\right) \right\}; \\ \text{s.t. } A = \left\{\boldsymbol{p} \in \Re^n : p_1 + \cdots + p_n = p\right\}, \end{cases} \tag{1.2}$$

where $\beta > 0$. The advantage of this rule is incorporating the variability into the allocation, and the decision-maker can adjust the weight of variability. In fact, the idea of incorporating the variability with the mean might be traced back to the Mean–Variance framework; see, for example, Steinbach (2001) and Landsman (2010). The mean–variance (MV) model uses the Mean–Variance risk measurement

$$\mathrm{MV}(X) = \mathbb{E}(X) + \beta \mathrm{Var}(X), \quad \beta \geq 0,$$

which is also known as the expected quadratic utility in finance literature.

(b) Tail risk. No tail risk has been considered in the allocation rule (1.1) or (1.2). In the premium literature, this problem has been well taken care of. Furman and Landsman (2006) used the tail variance risk (TVP) measure estimating the variability along the tails to compute the premium.

$$\mathrm{TVP}_q(X) = \mathrm{TCE}_q(X) + \beta \mathrm{TV}_q(X), \quad \beta \geq 0,$$

where

$$\mathrm{TCE}_q(X) = \mathbb{E}(X | X > \mathrm{VaR}_q(X)),$$

$$\mathrm{TV}_q(X) = \mathrm{Var}\left(X | X > \mathrm{VaR}_q(X)\right),$$

$\mathrm{VaR}_q := \inf\{x : x : F(x) \geq q\}$ is the qth quantile of X or VaR, and $F(x) := P(X \leq x)$ is the distribution of X.

Motivated by the above discussion, in this paper, we develop a new methodology for capital allocation, which considers the variability and tail risk of the loss function simultaneously. Denote by $S = \sum_{i=1}^n X_i$ the aggregate risk. We consider the following Tail Mean–Variance (TMV) model, which overcomes the limitations of allocation based on minimizing the loss function (1.1) or (1.2):

$$\begin{cases} \min_{\boldsymbol{p}\in A} \left\{ \mathbb{E}\left[\sum_{i=1}^{n} (X_i - p_i)^2 \,|S > \mathrm{VaR}_q(S)\right] \right. \\ \qquad \left. + \beta \mathrm{Var}\left(\sum_{i=1}^{n} (X_i - p_i)^2 \,|S > \mathrm{VaR}_q(S)\right)\right\}; \\ \text{s.t. } A = \left\{\boldsymbol{p} \in \Re^n : p_1 + \cdots + p_n = p\right\} \end{cases} \tag{1.3}$$

where $\beta > 0$, and $\mathrm{VaR}_q(S)$ is the qth quantile of S. This model could be considered as a natural extension of Model (1.2), as it reduces to Model (1.2) when $q = 0$. In this paper, we are making the following main contributions.

(a) Derive general optimal capital allocation formulas for the TMV model.
(b) Give explicit expressions for optimal allocations in the general framework of multivariate elliptical distributions.
(c) Provide asymptotic allocation formulas for multivariate regularly varying distributions.

The rest of this paper is organized as follows. In Section 2, we present general capital allocation formulas for arbitrary distributions based on the TMV model. Section 3 presents explicit formulas for the multivariate elliptical distributions. Asymptotic allocation formulas for the multivariate regularly varying distributions are given in Section 4. Real data from an insurance company is discussed in Section 5. In the last section we summarize the results, and present some discussions for future work.

2. General result

In this section, we provide a general capital allocation formula for the TMV model. The idea of proof is based on Kuhn–Tucker theory, which is similar to that of Ostaszewski and Xu (2012).

We first need the following lemma.

Lemma 2.1 (*Bertsekas, 1999*)**.** *Assume that f and h are twice continuously differentiable, and let $L(\boldsymbol{p}, \lambda) = f(\boldsymbol{p}) + \lambda h(\boldsymbol{p})$. If*

$$\nabla_{\boldsymbol{p}} L(\boldsymbol{p}^*, \lambda^*) = 0, \qquad \nabla_\lambda L(\boldsymbol{p}^*, \lambda^*) = 0,$$

and for all $\mathbf{y} \neq 0$ *with* $\left[\nabla_{\boldsymbol{p}} h(\boldsymbol{p}^*)\right]^T \mathbf{y} = 0$,

$$\mathbf{y}^\tau \nabla^2_{\boldsymbol{p}} L(\boldsymbol{p}^*, \lambda^*)\mathbf{y} > 0,$$

where ∇ is the differential operator and $\boldsymbol{x}^\tau$ is the transpose of $\boldsymbol{x}$, then $\boldsymbol{p}^*$ is a strict local minimum of f subject to $h(\boldsymbol{p}^*) = 0$.

Now, let us present the following result, which provides general optimal capital allocation formulas for the TMV model.

Theorem 2.2. *Let $\mathbf{X} = (X_1, \ldots, X_n)$, and assume $\boldsymbol{p}^* = (p_1^*, \ldots, p_n^*)$ is an optimal allocation solution to the TMV model. Then*

$$\boldsymbol{p}^* = \mathrm{A}^{-1}\boldsymbol{z},$$

where $\mathrm{A}^{-1} = (a_{ij})_{n\times n}$ is the inverse matrix of $\mathrm{A} = 8\beta\Sigma_S + 2\mathrm{I}_n$, where Σ_S is the conditional covariance matrix of $\left(\mathbf{X}|S > \mathrm{VaR}_q(S)\right)$, and $\boldsymbol{z}^T = (\lambda + \delta_1, \lambda + \delta_2, \ldots, \lambda + \delta_n)$ with

$$\delta_i = 4\beta \sum_{j=1}^{n} \sigma_{2,j,i} + 2\mu_i, \quad i = 1, \ldots, n,$$

where $\sigma_{2,j,i} = \mathrm{Cov}\left(X_j^2, X_i | S > \mathrm{VaR}_q(S)\right)$, and $\lambda = (p - \sum_{i=1}^n \sum_{j=1}^n a_{ij}\delta_j)/(\sum_{i=1}^n \sum_{j=1}^n a_{ij})$. More specifically,

$$p_i^* = \frac{p - \sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}\delta_l}{\sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}} \sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{n} a_{ij}\delta_j, \quad i = 1, \ldots, n.$$