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Valuing equity-linked death benefits in jump diffusion models

Hans U. Gerber [a,b], Elias S.W. Shiu [c], Hailiang Yang [a,*]

[a] Department of Statistics and Actuarial Science, The University of Hong Kong, Hong Kong

[b] Faculty of Business and Economics, University of Lausanne, CH-1015 Lausanne, Switzerland

[c] Department of Statistics and Actuarial Science, The University of Iowa, Iowa City, IA 52242-1409, USA

HIGHLIGHTS

- The stock price process is assumed to be the exponential of a jump diffusion.
- Results for exponential stopping of a Lévy process and the Wiener–Hopf factorization are employed.
- Options are exercised at the time of death.
- For a series of options, closed form formulas for their expected payoff are given.
- It is also discussed how barrier options can be used to model lapses and surrenders.

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ABSTRACT

The paper is motivated by the valuation problem of guaranteed minimum death benefits in various equity-linked products. At the time of death, a benefit payment is due. It may depend not only on the price of a stock or stock fund at that time, but also on prior prices. The problem is to calculate the expected discounted value of the benefit payment. Because the distribution of the time of death can be approximated by a combination of exponential distributions, it suffices to solve the problem for an exponentially distributed time of death. The stock price process is assumed to be the exponential of a Brownian motion plus an independent compound Poisson process whose upward and downward jumps are modeled by combinations (or mixtures) of exponential distributions. Results for exponential stopping of a Lévy process are used to derive a series of closed-form formulas for call, put, lookback, and barrier options, dynamic fund protection, and dynamic withdrawal benefit with guarantee. We also discuss how barrier options can be used to model lapses and surrenders.

1. Introduction

This paper is a continuation of Gerber et al. (2012). The motivation is the problem of valuing guaranteed minimum death benefits (GMDB) in various variable annuity and equity-indexed annuity contracts. Our goal is to present actuaries with a methodology that they can use to value and reserve for such guarantees.

In Gerber et al. (2012), the price of a stock or stock fund is modeled as a geometric Brownian motion, i.e., the price at time t is

$$S(t) = S(0)e^{X(t)}, \quad t \geq 0, \tag{1.1}$$

where $\{X(t); t \geq 0\}$ is a Brownian motion or Wiener process. In this paper, we generalize $\{X(t)\}$ as a jump diffusion, i.e., a Brownian motion plus an independent compound Poisson process. For actuaries, the use of compound Poisson processes can be traced back to the 1903 doctoral thesis of the Swedish actuary Filip

* Corresponding author. Tel.: +852 2857 8322; fax: +852 2858 9041.
E-mail addresses: hgerber@unil.ch (H.U. Gerber), elias-shiu@uiowa.edu (E.S.W. Shiu), hlyang@hku.hk (H. Yang).

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Lundberg. In Lundberg's collective risk theory, aggregate claims are modeled by compound Poisson processes. The family of compound Poisson processes is rich in that it is dense in the family of all Lévy processes, of which jump diffusions are special cases. Therefore, the addition of an independent Brownian motion was not a big step and, understandably, has not found wide publicity in the actuarial literature. Two early papers are Gerber (1970, 1972). The title of Dufresne and Gerber (1991b), "Risk theory for the compound Poisson process that is perturbed by diffusion", is indicative of the actuarial perception of the jump diffusion model.

In finance, the jump diffusion model is considered from a substantially different perspective. The economic significance of modeling stock price movements by geometric Brownian motion was recognized by Samuelson (1965). For modeling jumps in stock prices, Merton (1976) added an independent compound Poisson process to the Brownian motion. This was an important advance, as geometric Brownian motion models do not account for empirical facts such as heavy tails and volatility smiles. An excellent survey of jump diffusion models in finance can be found in Kou (2008), who and whose co-authors have done pioneering work in this field.

Jump diffusions are particularly tractable if the distribution of the jumps is a combination (or a mixture) of exponential distributions. In actuarial science, it is found that if the individual claim distribution is modeled as a combination (or a mixture) of exponential distributions, closed-form expressions for the probability of ruin, for the expected discounted dividends until ruin, and other quantities of interest can be readily determined. See, for example, Täcklind (1942), Dufresne and Gerber (1988, 1989, 1991a,b), Chan (1990), Gerber and Shiu (1998, 2005), Chan et al. (2006), Gerber et al. (2006), Avanzi and Gerber (2008), and so on.

Consider a variable annuity for a person now age x. There is a GMDB rider that guarantees the following payment to his estate when he dies,

$$\max(S(T_x), K), \tag{1.2}$$

where T_x is the time-until-death random variable and K is the guaranteed minimum amount. Since

$$\max(S(T_x), K) = S(T_x) + [K - S(T_x)]_+, \tag{1.3}$$

the problem of valuing the guarantee becomes the problem of valuing a K-strike put option that is exercised at the time of death. Because policy surrenders or lapses should also be incorporated in the valuation model, the problem is to determine the following expectation:

$$\mathrm{E}[e^{-\delta T_x} e^{-\theta T_x} \pi(S(0)e^{M(T_x)})[K - S(T_x)]_+]. \tag{1.4}$$

Here, δ denotes a continuously compounded valuation interest rate (valuation force of interest). $M(T_x)$ is the maximum of $\{X(t)\}$ up to T_x; because of (1.1), $S(0)e^{M(T_x)}$ is the maximum price of a unit of the stock fund between time 0 and T_x. The function $\pi(s)$ is 1 for $s \le S(0)$, and it is a nonnegative and nonincreasing function of s for $s > S(0)$; the function $\pi(s)$ is to capture the phenomenon that the higher is the stock price, the less valuable is the guarantee (put option), and hence the higher is the tendency for policyholders to surrender their policies. The exponential function $e^{-\theta t}$, with θ being a positive constant, models that independent of the stock fund performance, a constant proportion of remaining policies will lapse in each subsequent time period. The product of the two factors, $e^{-\theta t}\pi(S(0)e^{M(t)})$, gives the fraction of in-force policies at time t.

By considering the valuation force of interest to be $(\delta + \theta)$, we can ignore the factor $e^{-\theta T_x}$ in (1.4). Thus we are motivated to examine expectations of the form

$$\mathrm{E}[e^{-\delta T_x} g_{T_x}(S)], \tag{1.5}$$

where $g_t(S)$ is a *functional* of the stock price process up to time t. For the case

$$g_t(S) = \pi(S(0)e^{M(t)})[K - S(t)]_+, \tag{1.6}$$

we seem to need to find the joint probability density function (pdf) $f_{X(T_x),M(T_x)}(x, y)$ for evaluating (1.5).

Here is a summary of our approach. (I) The distribution of the positive random variable T_x can be approximated by linear combinations of exponential distributions. Then, under the assumption that T_x is independent of the stock price process $\{S(t)\}$, the problem of approximating the expectation (1.5) reduces to that of evaluating

$$\mathrm{E}[e^{-\delta\tau} g_\tau(S)], \tag{1.7}$$

where τ is an arbitrary exponential random variable independent of $\{S(t)\}$. (II) We can use the factorization,

$$\mathrm{E}[e^{-\delta\tau} g_\tau(S)] = \mathrm{E}[e^{-\delta\tau}]\mathrm{E}^*[g_\tau(S)] = \frac{\lambda}{\lambda+\delta}\mathrm{E}^*[g_\tau(S)], \tag{1.8}$$

to take care of the discount factor. Here, λ is the parameter of τ, i.e., $\mathrm{E}[\tau] = 1/\lambda$. The asterisk signifies that the parameter of τ is changed to $\lambda + \delta$, i.e., $\mathrm{E}^*[\tau] = 1/(\lambda + \delta)$. Hence, this paper will derive formulas for

$$\mathrm{E}[g_\tau(S)], \tag{1.9}$$

not (1.7). (III) Let $M(\tau)$ denote the running maximum of the Lévy process $\{X(t)\}$ up to time τ. The random variables $M(\tau)$ and $[X(\tau) - M(\tau)]$ are independent. Hence the joint pdf of $X(\tau)$ and $M(\tau)$ can be factorized,

$$f_{X(\tau),M(\tau)}(x, y) = f_{M(\tau)}(y) \times f_{X(\tau)-M(\tau)}(x - y), \quad \max(x, 0) \le y. \tag{1.10}$$

To determine the two pdfs on the right-hand side (RHS) of (1.10), we find their moment generating functions (mgf) by means of the identity

$$\mathrm{E}[e^{zX(\tau)}] = \mathrm{E}[e^{zM(\tau)}] \times \mathrm{E}[e^{z[X(\tau)-M(\tau)]}]. \tag{1.11}$$

Details of this important step are given in Section 3.

For readers interested in Laplace transforms, we note the following two facts. Consider the expectation

$$\mathrm{E}[g_t(S)] \tag{1.12}$$

as a function of t, $t \ge 0$; the Laplace transform of (1.12) with respect to the parameter λ is $\mathrm{E}[g_\tau(S)]/\lambda$. Consider the surplus process defined by

$$U(t) = u - X(t), \quad t \ge 0;$$

the Laplace transform of the time of ruin random variable with respect to the parameter λ is the probability $\Pr(M(\tau) \ge u)$.

We consider options that are exercised at time τ. In Sections 4–8, we derive formulas for valuing various call, put, lookback, and barrier options. Section 9 values "dynamic fund protection" when the guarantee is effective until time τ. Section 10 considers the dual concept of "dynamic withdrawal benefit" and values a put option on the residual account value exercised at time τ.

2. Exponential stopping of a Lévy process

In this section we set up a general framework. More specific results will be given in subsequent sections.